Assessment Guide and Practice Questions

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1 Assessment criteria

1.1 Assessment criteria for Microeconomics 1

My half of Microeconomics 1 is worth 50% of both the December and May exams. Your final mark is based on the maximum of your December and May exam marks.

My part of the class and degree exams have an identical format and marking scheme, approximately as follows (not counting the bonus questions). You will be rated as no/almost/ok/good/excellent (i.e. 0 to 4) on the following learning outcomes:

- Formulating a model. Excellence here means the absence of important mistakes.
- Breadth of technique: Walras law, dynamic programming, the envelope theorem, convex analysis, the first welfare theorem, etc. Excellence here usually means applying four techniques.
- Depth of technique: using a technique in an unusual way, combining several techniques to deduce something, or a clever piece of logic. For example, proving that there is at most one equilibrium in a particular model by combining the first welfare theorem with symmetry of all households. Obviously, depth requires at least some breadth, so this is correlated with the breadth learning outcome. Excellence here means that the assumptions, conclusions, and the logic from one to the other are clearly expressed.

There is no precise system for determining marks, but a linear regression reveals that marks will usually be close to $45.5 + 2.7m_1 + 3.8m_2 + 4.4m_3$, where m_i is the mark on learning outcome i on the $0 - 4$ scale. This formula is less accurate at both extremes – all marks between 0 and 100 are possible.

Note that exams have become longer and more difficult in recent years to give multiple opportunities to show the third learning outcome. Thus, it has become easier to get a high mark.

1.2 Assessment criteria for Mathematical Microeconomics 1

You will sit two exams in December, and another two exams in May, all of which are three hours (12 hours total). Each pair of exams consists of the Microeconomics 1 exam and Part B of the Advanced Mathematical Economics exam. The Microeconomics 1 exams are marked exactly the same way as for Microeconomics 1 students. These exams count for two thirds of your mark.

The mathematics exams count for the remaining third of your mark. Only your best exam (December or May) will be used. The mathematics exam will be marked against four criteria:

- Fundamentals (48 points). If you pass any other criterion below, you will automatically get all of these points.
- Definitions (10 points). Full marks will be awarded if you can reproduce four mathematical definitions.
- Reformulation (10 points). Writing a proof usually involves restating the question in a form that is convenient for writing a proof. For example, you might need to expand a definition, reformulate as a contrapositive, split up an if and only if into the two directions, and so on. Full marks will be awarded if you do useful reformulations for two questions.
- Deduction (32 points). This mark is determined by the quantity and quality of "snippets of logic." I almost never give a mark in the 0-4 range. A mark in the range of 5-9 corresponds to being able to prove a simple theorem (with clear reasoning) or providing an example or counterexample. For example, proving that the interior and boundary of a set is disjoint would lead to a mark in this range. A mark in the range of 10-15 involves being able to prove two simple theorems. A mark in the range of 16-20 involves being able to prove one intermediate theorem (requiring many steps or integrating several ideas). A mark in the range of 21-32 involves writing a proof with a degree of mathematical creativity in combining ideas (such as 24.B.vii) or two intermediate theorems. It is ok to skip "easy" steps, just answer part of a question, and/or use theorems from lectures or homework. What matters is how you put it all together, and any logical manoeuvres or creative ideas you add in.

1.3 Assessment criteria for Advanced Mathematical Economics (undergraduate)

There are two versions of this course – undergraduate (ECNM10085) and postgraduate (ECNM11072). What follows here is the assessment for the undergraduate version. Homework is worth 10% of the mark. It is marked on effort only – if you attempt a majority of questions, you will score full marks. Homework can be submitted at the start of the lecture, or electronically via Learn. If you choose electronic submission, you can either scan, photograph, or type your homework. You might find Microsoft Lens convenient for this. You must submit at least 6 of the 9 problem sets. Otherwise, you will be penalised 2% for each additional problem set you missed. You will receive feedback on your homework during tutorials. I recommend that students ask each other for help, and also ask for help during Tutorials and also on Piazza.

One-semester visiting students can do an optional project about global warming. If the project mark is higher than the exam mark, then the final mark will be calculated based on 10% homework + 45% project + 45% exam. If the project mark is lower than the exam mark, then the final mark will be calculated as 10% homework + 90% exam.

Exams are worth 90% of the mark. Full-year students can take both the December and May exams, whereas one-semester visiting students can only take the December exam. The better exam mark will be used to calculate the course mark. Exams are marked against the following criteria:

- Fundamentals (45 points). Students automatically get full marks on this criterion if they earn any marks on any of the other criteria. Otherwise, the mark on this criterion reflects the basic knowledge of mathematics and economics demonstrated by the student.

- Model formulation (10 points).

- Applying theorems to models (10 points). Full marks usually involves applying three techniques correctly.

- Proving mathematical theorems (35 points). This is by far the hardest learning outcome, and mostly corresponds to the questions in Part B. This mark is determined by the quantity and quality of "snippets of logic." I almost never give a mark in the 0-4 range. A mark in the range of 5-9 corresponds to being able to prove a simple theorem (with clear reasoning) or providing an example or counterexample. For example, proving that the interior and boundary of a set is disjoint would lead to a mark in this range. A mark in the range of 10-15 involves being able to prove two simple theorems. A mark in the range of 16-20 involves being able to prove one intermediate theorem (requiring many steps or integrating several ideas). A mark in the range of 21-35 involves writing a proof with a degree of mathematical creativity in combining ideas (such as 24.B.vii) or two intermediate theorems.

For example, if you did all of your homework, made only minor mistakes in Part A, and were able to answer two of the easier Part B questions well, then your mark would be approximately, $10 + 0.9(45 + 9 + 9 + 15) = 80$.

I strongly recommend that students attempt Part A first, which is primarily about the model formulation and theorem application criteria. Part B is primarily about proving mathematical theorems.

I rarely set exam questions that appeared in the lecture notes – the class is about writing proofs, not memorising them. All material from the sections that I cover are examinable, unless I say it is not. For example, I said we skipped quasi-convexity in Appendix D. At the end of semester, I compile a comprehensive list of what is examinable. You can see the list from previous years by viewing "last years' course materials". The harder questions are designed to separate outstanding students (who deserve marks in the 90s) from excellent students. I would like it to be very transparent to students what they need to do to earn outstanding marks.

I believe all students have the potential to be outstanding, despite the fact that some students bring advantages with them at the beginning. I am doing my best to figure out how to bring out the best in all students. Students with low marks in the rest of their degree often achieve excellent results in my course. It would like this to happen even more often.

Questions 20, 21, 24, 27, 29, 31, 34 are specifically written for Advanced Mathematical Economics. All other questions are from Microeconomics 1, which covers different material. Many questions are based on material that was taught, but only in passing. These questions would be examinable as more advanced questions (and hence would attract a bigger reward if correctly answered.) These questions are *not* examinable, as they are based on ideas that were not covered in lectures at all:

1 (v), (vi), (vii).

2 –

3 (ii), (vii), (viii).

4 (iv), (vi), (vii).

5 (vi).

6 (ii), (vi), (vii).

7 (ii), (vi).

8 (ii), (iv).

9 (v).

10 (ii), (v), (vi).

11 (vi).

12 (iii), (v).

13 (ii), (vi), (vii).

14 (vi), (vii).

15 (ii), (v), (vi).

16 (iv), (vi).

17 (ii), (vi).

18 (ii), (vi), (vii).

19 (vi).

20 –

21 –

22 (iv), (v), (vi).

23 (ii), (v), (vi), (vii).

24 –

25 (ii), (iv), (v).

26 (iv), (v), (vi).

27 –

28 (ii), (vi).

29 –

30 (ii), (iii), (iv), (v).

31 –

32 (ii), (iv), (v).

33 (ii), (v), (vi).

34 –

1.4 Assessment criteria for Advanced Mathematical Economics (postgraduate)

The postgraduate version of Advanced Mathematical Economics is taken by CPD students and PhD students. It is different from Mathematical Microeconomics 1 (see above). The assessment is similar to the undergraduate version of Advanced Mathematical Economics, except:

- The weekly homework is not formally assessed.
- You must do the project, which is worth 20%.
- The exam(s) are only worth 80%.

As in the undergraduate version of the course, you can either enrol to take the course over one semester or the whole year.

2 Advice for Answering Exam Questions

2.1 Generic Advice

- There is no need to add extra complications into the model. For example, if the question does not mention time, then there is no need to put multiple time periods into the model.
- If you can't figure out the answer, don't pretend you know it. It's better to explain what you are confused about – a well written statement of confusion can illustrate that you know the material very well, and give you a very good mark.

- Even if you misformulate your model, this shouldn't stop you from answering subsequent parts. But if the model then seems inconsistent with the question (e.g. the question asks "show real wages are higher" when in your model, this is not true) then please do not try to prove the impossible. Instead, please either explain why the question is inconsistent, or if you're not sure, explain why you are stuck and can't complete your argument.

- Students often incorrectly identify the envelope formula as a first-order condition. It's not. First-order conditions are about optimal choices. If you are differentiating with respect to prices, you are not doing a first-order condition, because in competitive markets, nobody can choose prices.

- Students often confuse value functions and objective functions. For example, students often (mistakenly) write that a firm's first-order conditions with respect to the number of workers involves differentiating the profit function (rather than the firm's objective function) with respect to its labour input. But the profit function is a function of prices, not quantities, so it makes no sense to differentiate it with respect to a quantity.

- You can introduce assumptions at any point in the paper. For example, if you discover in part (iv) that you need to assume that the production function is concave, then you can write that assumption in your answer to part (iv). You do not need to revise your answer to part (i).

- In proofs and calculations, please write with complete grammatical sentences, including punctuation.

- In proofs, be careful to distinguish between "there exists" and "for all".

- In proofs, be careful to distinguish between set membership ($\in$) and subsets ($\subseteq$).

2.2 A Checklist

A mark below 50% means something important was missing from your model. For example, you might have had two different markets with the same price, or a firm buying something (like a wholesale good) without using it in production. Here is a check-list of important ingredients of every economic model:

- Any notation is fine, but you must define it.

- When writing down the agents' optimisation problems, you should always write the choice variables under the *max*.

- In competitive models, agents only choose quantities, not prices.

- Every market has one (and only one) price. For example, labour markets have only one price if all types of labour are equally valued (by buyers and sellers). On the other hand, if workers have preferences over their profession, or firms value some workers above others, then these are separate markets and have separate prices.

- Every cost should also have a corresponding benefit (and vice versa). There are exceptions to this rule (e.g. inelastic labour supply), but think carefully about this.
- Every market should have a market-clearing condition. Thus, there are always an equal number of prices and market-clearing equations. It also means you need to define notation for both supply and demand. (In the sample solutions, I typically write firm decisions in upper case, and household decisions in lower case.)

2.3 Notation

Notation for partial derivatives: there are many common (correct) ways to write partial derivatives, including

$$\frac{\partial}{\partial x} f(x, y) \tag{1}$$
$$\frac{\partial f(x, y)}{\partial x} \tag{2}$$
$$f_x(x, y) \tag{3}$$
$$f_1(x, y) \tag{4}$$
$$D_x f(x, y) \tag{5}$$
$$D_1 f(x, y) \tag{6}$$
$$\nabla_x f(x, y) \tag{7}$$
$$\nabla_1 f(x, y). \tag{8}$$

Writing

$$f'_x(x, y) \tag{9}$$

is *not* standard, so I suggest you avoid it. (It is unambiguous though, so it wouldn't lose you marks in my exams.)

The notation $f'(x, y)$ or $Df(x, y)$ or $\nabla f(x, y)$ does *not* represent a partial derivative, but rather the *total* derivative, i.e. the vector (or matrix) of partial derivatives of f. Please don't write this if you mean a partial derivative.

3 Practice Questions

1: Micro 1, mock exam

Consider a pure-exchange economy in which all goods are produced from oil by home production over 2 time periods. Only oil is traded. There are two households and two oil deposit sites of size 1. The first site is owned by household A, and oil can be extracted from it at any rate over the 2 periods. The second site is owned by household B, but oil production is only possible in the second period. Both households have the same preferences, which are impatient discounted utility with the same per-period utility function which is strictly concave.

(i) Define an equilibrium in this economy.

(ii) Write down the egalitarian social planner's problem (i.e. assuming that the social planner puts equal weight on the households.) What allocation would she choose?

(iii) In equilibrium, how do oil prices change over time?

(iv) In equilibrium, which household is better off? Explain.

(v) Suppose there is a bubble, in the sense that in the last period, oil prices are too high and there is excess supply of oil in the last period. What would happen in the first period? (Hint: Walras' law.)

(vi) * Which assumptions above about the households' utility are relevant for Debreu's theorem about additively separable preferences? Which assumptions go beyond the conclusion of Debreu's theorem?

(vii) * What additional assumptions are needed to ensure existence of equilibria in this economy?

2: Micro 1, mock exam

The cashew tree is native to the Amazon forest in Brazil, its fruit is about the same size as an apple. The juice of the flesh of the fruit is popular in Brazil (along with açaí, acerola, guava, mango, papaya, and many others... but ignore those!) Each fruit has enough juice to fill a single cup. Each fruit also contains a single seed, which when toasted becomes the cashew nut which is popular all over the world.

(i) The firm chooses how many cashew fruits to grow (which requires labour), and then sells the juice and nuts. Assume that no work is required to extract the juice and nuts – only growing requires labour. Write down the firm's profit function.

(ii) Write down the firm's cost function. Hint: you will need two quantities in the state variable (as well as factor prices).

(iii) There are several identical households that supply labour and consume cashews and cashew juice and hold equal shares in the cashew firm. Write down a general equilibrium model of the economy.

(iv) Write down a utility function for the households consistent with the idea that households enjoy cashew nuts more than cashew juice. What can you say about equilibrium prices in this case?

(v) Does the firm have increasing marginal cost in both products?

(vi) Sketch a graph of the firm's marginal cost of producing cashew juice, holding fixed the number of cashew nuts being produced at 3.

3: Micro 1, December 2012

A farm produces food from labour. However, the farm does not have a distribution network, so it can not sell the food directly to the households. Rather, it must sell the food to a supermarket at a wholesale price, which then resells to households at a retail price. The supermarket buys food and labour, which it uses to resell the food. Some food might get wasted; more labour means less food gets wasted. All households are identical, and supply labour to both firms.

(i) Formulate an economy by writing down the households' and firms' value functions, and the market clearing conditions. Focus attention on symmetric equilibria, i.e. in which all households make the same decisions. (Hint: you might find it helpful to consider the wholesale food a completely separate good. Don't forget profits.)

(ii) Select a constraint which may be dropped by Walras' law.

(iii) Suggest how an endogenous variable may be eliminated, since inflation of all prices by an equal factor does not affect decisions.

(iv) Show that the supermarket's profit function is convex. (Hint, you may use the following theorem from class: Suppose V is the upper envelope of convex functions, i.e. $V(a) = \max_b v(a, b)$ where $v(\cdot, b)$ is a convex function for each b. Then V is convex.)

(v) Show that the supermarket responds to a wholesale price increase by buying less.

(vi) There have been protests recently that the (equilibrium) retail price is much higher than the wholesale price, which the households feel is grossly unfair. They propose introducing a profit tax of 50% to be redistributed equally among households, a price markup ceiling of 10%, and a minimum wage increase of 20%. Would this policy make the households better off (under standard assumptions, like increasing utility functions)?

(vii) * Prove that the supermarket's policy is continuous if its production function is strictly concave. You may assume that the supermarket only has space to accommodate a maximum number of workers and amount of food.

(viii) * To prove existence of equilibrium using Brouwer's fixed point theorem, it is important that the set of possible prices are compact. Explain why this is important, and how to accommodate this requirement.

4: Micro 1, December 2012

Sackman, Erickson, and Grant (1968) conducted an experiment on computer programmers, which they published in the Communications of the Association of Computing Machinery. They summarised their findings with the following poem:

> When a programmer is good,
> He is very, very good,
> But when he is bad,
> He is horrid.

Even though the programmers were quite experienced, there was very wide disparity in their abilities. They found the best programmer writes their code about 20 times more quickly than the worst programmer. They debug it 28 times more quickly, the final code runs about 10 times faster, and so on. Follow-up studies report similar disparities, and it has become conventional wisdom that the best computer programmers are about 10 times more productive than the median.

Suppose there is a mediocre and a brilliant computer programmer. Assume that one hour of work by the brilliant programmer is a perfect substitute for ten hours of work by the mediocre programmer. The households are otherwise identical and hold equal shares in the firm.

(i) Write down a model of this economy, and define a general equilibrium for it.

(ii) Show that in every equilibrium in which both programmers are hired, the brilliant programmer's wage is ten times higher than the mediocre programmer's wage.

(iii) Show that in every equilibrium, the brilliant programmer is better off than the mediocre programmer.

(iv) Depending on the preferences of the households, the brilliant programmer might work longer or shorter hours. Draw the indifference curves in a way that indicates the brilliant programmer working *less* than the mediocre programmer.

(v) Some people think that the problem is that mediocre programmers are lazy, and they just need some extra incentives to work hard. In the context of your model, would giving the programmers stock options, 100% bonus pay upon project completion and hiring a masseuse and celebrity chef make everyone better off?

(vi) The mediocre programmer has another more Machiavellian proposal for increasing productivity. He proposes asking the government to issue a large lump-sum tax on the brilliant programmer, which will force her to work long hours to repay her (government-imposed) debt. The mediocre programmer further proposes the he receive the taxes. Would this proposal work?

(vii) * Discuss the problems with proving existence in this economy.

5: Micro 1, May 2013

We eat about 300 billion apples every year, but most of these apples can not be eaten directly from the tree. The problem is that apples only ripen in Autumn, and apples consumed at other times must be stored. On the other hand, lettuce may be grown in all seasons, so it is never necessary to store it. Henceforth, assume it is non-storable.

Suppose there are just two seasons (Autumn and Spring) and two foods (lettuces and apples). Farmers are endowed with apples in Autumn, and lettuce in equal quantities in both Autumn and Spring. There is a storage firm (owned by the farmers) that can refrigerate apples until the Spring. The storage technology does not require any labour or other resources to operate. However, as they store more fruit, they become less effective and an increasing fraction of apples go bad.

(i) Define a general equilibrium in this setting, focusing attention on symmetric equilibria in which all farmers make the same decisions as each other.

(ii) Is it possible to normalise apples prices to 1?

(iii) Show that if the storage technology is perfect, then apples prices are equal in both seasons.

(iv) Show if the storage technology involves some spoilage, that apples are more expensive in Spring than Autumn.

(v) Suppose that the farmers' preferences have a discounted utility representation. (i.e. Time separable preferences that can be written in an additively separable fashion, with per-period utility functions being identical.) Moreover, assume that the farmers have decreasing marginal utility in apple and lettuce consumption. (a) Write the farmers' first-order conditions, (b) show that the farmers consume more apples in Autumn than Spring, and (c) write the farmer's problem using a Bellman equation.

(vi) Now suppose that one farmer is extra productive, and has double the endowments of all of the other farmers. The other farmers have a smaller endowment so that the aggregate endowments are identical. Think about the prices in the following scenarios:

   (a) The original symmetric equilibrium.
   (b) The new equilibrium (with the extra productive farmer).
   (c) A new equilibrium (with the extra productive farmer) in which the productive farmer is taxed so that the equilibrium allocation is the same as in (a).

   Do any of these scenarios share the same equilibrium prices?

(vii) Show that the farmers' second-period value function is concave and ** differentiable.

6: Micro 1, May 2013

Suppose there are two countries of equal population. However, the big country has twice the amount of land, so that each household located there has twice the land endowment of households in the small country. Each country has an agricultural firm that transforms labour and land into food. Food can be traded on the international market. However, labour and land are more complicated. Each firm is owned equally by the citizens of its own country, and can only grow food on its own country's land. We say that workers migrate if they work for the other country's firm, although we assume that migration is costless.

(i) Write down a general equilibrium model of the labour, food and land markets. (Hint: treat labour and food as unified international markets, but land as national markets.)

(ii) Suppose that at some (out-of-equilibrium) prices, the food and labour markets clear, but there is excess demand of the small country's land. What does Walras' law say about the market for the large country's land?

(iii) Show that the small country's firm's profit function is convex in prices.

(iv) Show that if wages increase, the small country decreases its demand for labour.

(v) Show that if the production technology has constant returns to scale, and leisure is a normal good, then there is some migration from the small to the big country. (Hint: functions that are homogeneous of degree 1, i.e. satisfy the property that $f(tx, ty) = tf(x, y)$, also have the property that $f_x(2x, 2y) = f_x(x, y)$ for all (x, y).)

(vi) Suppose the two countries plan to federalise into a free-trade zone (like the EU). They are worried about social tensions arising from the inequality of the people from the two countries. Devise a lump-sum tax scheme that creates perfect equality.

(vii) * Suppose that households are constrained to work in one country only (of their choice). Discuss how this possibility impedes application of the Brouwer's fixed point theorem to establish existence of equilibria.

7: Micro 1, December 2013

US comedian Lewis Black has the following to say about solar energy:

> If you ask your congressman why, he'll say "Because it's hard. It's really hard. Makes me want to go poopie." You know why we don't have solar energy? It's because the sun goes away each day, and it doesn't tell us where it's going!

Two countries are endowed with some electricity during the day time. However, they are located on opposite sides of the world, so when it is day time in one country, it is night time in the other. Electricity is non-storable, so the only way to consume electricity at night is to import electricity from the other country. A portion of the electricity is lost in transportation; the fraction lost increases as the amount of electricity transported increases.

Apart from this, the countries are identical: there is one household in each country, they share the same preferences and endowments, and the household in each country owns its own electricity exporter. You may assume preferences are additively separable across time, and they value electricity consumption equally during the day and night with decreasing marginal utility.

(i) Write down a general equilibrium model of this economy for one 24-hour period consisting of one night and day in each country. (Hint: treat electricity in different countries and different times as separate markets.)

(ii) It is possible to eliminate equilibrium variables and conditions using (i) price normalisation and (ii) Walras' law. Provide specific examples of how each of these may be done in the context of your model.

(iii) Suppose that both distributors discover a perfect transportation technology that prevents any electricity from being lost in transportation. In this case, show that both countries have the same sequence of electricity prices.

(iv) Show that if the distributors have a perfect transportation (as above), then the prices are the same. (Hint: look at the households' first-order conditions, and check the market clearing conditions.)

(v) Consider the proposal of taxing electricity consumption to subsidise electricity distributors to compensate them for the wasted energy lost. Would this proposal make everybody better off?

(vi) Again, suppose that there is a perfect transportation technology (see above). Consider the proposal of one country to invade the other, and to impose a new lump-sum tax on the victim country's household. The booty is distributed to the invading country's household. Does this make the invading household better off?

8: Micro 1, December 2013

Suppose there are two types of people: words people and numbers people. A medicine factory hires workers into two professions: marketing and engineering. Both types of people can do both types of jobs, but words people are better at marketing, and numbers people are better at engineering. Specifically, one hour of a words person's time spent on marketing is equivalent to two hours of a numbers person's time spent on marketing, and vice versa. Both types of people have the same preferences, and are indiffferent between both professions – they just take the best wage they can find. Everybody knows what type of person they are trading with.

(i) Define an equilibrium for this economy.

(ii) Suppose there is excess demand for both types of labour, i.e. at market prices, the firm demands more labour than the workers are willing to supply. Does this mean that there is also excess demand for medicine?

(iii) The factory has to make two types of choices: how many workers of each type to hire, and how to allocate them to professions.

  (a) Define the firm's output function as the maximum amount of medicine the firm can produce with given labour inputs.

  (b) Write down a Bellman equation for the factory relating the firm's cost function to the firm's output function.

  (c) Show that the firm's cost function is concave with respect to wages.

  (d) Show that if the market wage of numbers people increases, then the firm finds it optimal to meet its production target by hiring fewer numbers people and more words people.

(iv) Suppose the Words Union has an agreement which guarantees a maximum number of hours for words people only, and that this makes the words people better off. The Numbers Union proposes offering the Words Union a deal: it would tax numbers workers a little bit, and give those taxes to words workers. In return, the Words Union would abandon its maximum hours policy. Is it possible that both unions would agree to this deal?

(v) * Prove that the cost function is differentiable with respect to wages.

9: Micro 1, December 2013

A child care centre provides any number of hours of care to several households using two types of labour: babysitters and cleaners. Both types of labour are necessary for production – if either is zero, then no care can be provided. Households can simultaneously supply labour of both types. Households are also endowed with divisible houses, which they can exchange.

(i) Define the concept of a symmetric equilibrium for this economy, in which each household makes the same choice.

(ii) Suppose at all equilibrium allocations, the households have a higher marginal utility loss of cleaning than babysitting. Show that in every equilibrium, the cleaning wage is higher than the babysitting wage.

(iii) Suppose that the firm's production function is not concave. Does this imply that the profit function is not convex in prices?

(iv) Suppose that workers must specialise in at most one profession, babysitting or cleaning. (This isn't a government restriction, just a difficulty of working in these professions.) Are all equilibria efficient? Specifically, is it the case that every equilibrium in this environment is Pareto undominated by every feasible allocation in this environment?

(v) * As in the previous part, suppose that workers must specialise in at most one profession, babysitting or cleaning. Can every efficient allocation in this environment be implemented using lump-sum taxes?

10: Micro 1, May 2014

Suppose there are two rural districts that share an identical agricultural technology for transforming water into food. In the first year, households in both districts are endowed with the same amount of water, which they sell to farms. In the second year, one district suffers a perfectly predictable drought and has no water endowment. Households only directly consume food, and only hold shares in local farms. There are no import/export or migration costs, but food and water are non-storable.

(i) Write down a competitive general equilibrium model of the economy. You may assume households' preferences can be represented by an additively separable utility function.

(ii) Suppose that some protesters succeed in lowering the price of water in the second period, which leads to excess demand of water in the second period. According to Walras' law, what other consequences would this non-equilibrium behaviour have?

(iii) Show that each household has a decreasing marginal value of saving for the second year, provided that the household has a decreasing marginal utility of consumption. (Hint: this involves formulating the value of savings.)

(iv) Show that each household consumes less during the drought.

(v) The government would like to compensate the drought-striken district. Either devise a lump-sum tax policy that would implement smooth (constant) consumption over time for all households, or prove that this task is impossible.

(vi) * Write down a function that has the following property: a price vector is a fixed point of that function if and only if there exists an equilibrium with that price vector. Your function should never lead to negative prices. (You may make use of the excess demand function without defining it explicitly.)

11: Micro 1, May 2014

Individuals are endowed with one unit of human capital and time. In the first year, individuals divide their time between accumulating human capital (through self-study), labour, and leisure. In the second year, the individuals divide their time between labour and leisure only. A firm produces a consumption good in each year using labour. The contribution of each hour of work to production is proportional to the worker's human capital.

(i) Write down a perfectly competetive model for this market. You may assume the households have additively separable utility, with stationary flow utility. (Hint: the human capital production function should have decreasing marginal product.)

(ii) Is it possible for the price of consumption in the first period to be 1?

(iii) Write down a value function for the start of the second year. (Hint: the state variable includes human capital, savings, and the prices in the second year.)

(iv) Derive the marginal value of (a) human capital and (b) savings.

(v) The government thinks that it's wasteful for everybody to become educated. It proposes a tax on labour earnings in the second year to encourage more labour to be supplied in the first year. Could such a policy be Pareto-improving?

(vi) * Informally discuss whether there are any asymmetric equilibria (e.g. in which some people choose to become well-educated, but others do not.)

12: Micro 1, December 2014

A factory produces appliances using labour and waste disposal services. Households supply labour and waste disposal. Households are endowed with small or large gardens, where they can dispose of waste. Assume that households do not suffer from storing waste in their gardens, and that gardens are not traded (or at least, not directly).

(i) Write down a competitive model of the labour, appliance, and waste disposal markets.

(ii) Show that in every equilibrium, all households' gardens are filled to capacity with waste.

(iii) Show that if leisure is a normal good, then households with bigger gardens work less.

(iv) Show that if the price of waste disposal increases, then firms will generate less waste.

(v) Suppose the government wants to decrease the amount of waste stored in gardens. Is there a lump-sum tax scheme that would work?

(vi) * Under what conditions would the households have a unique optimal labour, appliance and waste storage choice?

(vii) * Prove that if all prices are greater than zero, and that households can work at most 24 hours per day, then the budget set (i.e. the set of affordable feasible choices) is compact.

13: Micro 1, December 2014

As the earth's population grows, an important question is how future inhabitants will be able to feed themselves, and whether this will lead to inter-generational inequality. Suppose there are two generations (X and Y) of equal size. Generation X lives for two time periods, but Generation Y only lives in the second time period. This means that the population is higher in the second period.

Farms produce food using land and labour. Only Generation X is endowed with land, which it can supply to the market. Generation X households hold all of the shares in the farms. Both generations can supply labour and consume food. Households do not benefit from occupying land (but can gain wealth from renting out the land). Generation X has stationary time-separable preferences, and its per-period utility function is the same as Generation Y's.

(i) Write down a competitive general equilibrium model of this economy.

(ii) Suppose that if the prices in all markets (labour, land, and food) do not increase over time, that there is excess demand of labour, land, and food in the second period. Does this imply that there is excess supply in all markets in the first period?

(iii) For this part, focus attention on equilibria in which food output is higher in the second period. Show that in every such equilibrium, real wages (i.e. wages divided by food prices) are lower in the second period.

(iv) Write down Generation X's value of holding money in the second period. (Hint: this should be a function of money and second period food prices and wages.)

(v) Reformulate Generation X's problem by using the value function from (iv) twice, i.e. the household should choose how to allocate money between the two periods. How the money is spent in each period should be buried inside the value function.

(vi) Generation Y protestors would like to eat more and work less, so they propose confiscating land from Generation X at the start of period 2, and giving it to Generation Y. Can such a policy make Generation Y better off? Would the proposal lead Generation Y to eat more and work less?

(vii) * The proof of existence of equilibrium relies on applying Brouwer's fixed point theorem, which requires a set to be convex (among other things). Economically speaking, which set is convex? Is this assumption usually met?

(viii) * Holding prices fixed, consider a sequence of optimal labour supply and consumption choices, where the expenditure decreases to 1. Does this sequence have a convergent subsequence (using the Euclidean metric)?

14: Micro 1, May 2015

Suppose there are two occupations, nursing and cleaning, and that individuals must select only one occupation to work in each year. Cleaning is easy to learn, but nurses with one year of experience become more productive. There are two years in the economy. Hospitals hire nurses and cleaners to provide medical services, and share their profits equally among the population. Individuals consume medical services.

(i) Write down a competitive model of the nursing and cleaning markets across the two years. (Hint: there are no symmetric equilibria, so you will need to accommodate identical households taking different decisions.)

(ii) Write down a formula for the value of savings and nursing experience in the second year.

(iii) Reformulate the year-one households' problem using the value function from the previous part.

(iv) What is the marginal value of nursing experience if the individual finds it optimal to do cleaning in the second year?

(v) Argue informally that nurses have lower wages than cleaners in the first year.

(vi) Are competitive equilibria Pareto efficient in this economy? (Hint: list all the differences from pure-exchange economies where we proved the first-welfare theorem, and informally discuss whether these are important.)

(vii) * Is the excess demand function continuous?

(viii) ** Is the household's feasiable choice set compact, assuming all prices are strictly greater than zero?

15: Micro 1, May 2015

Suppose there are two schools that hire workers to teach. One school is twice as productive as the other – i.e. for the same amount of input, it produces double the output. Households supply labour and consume education.

(i) Write down a competitive model of this economy.

(ii) Suppose at prevailing prices, there is excess supply of teachers. What does this imply about the supply of education?

(iii) Prove that the "good" (more productive) school hires more teachers than the "bad" school.

(iv) Prove that if wages increase, then schools provide less education.

(v) Suppose that the government imposes lump-sum taxes on half of the population, and transfers these to the other half equally. Moreover suppose that education and leisure are normal goods, and that this policy causes real wages to increase. What happens to each household's education choices? Hint: the Slustky equation is:

$$\underbrace{\frac{\partial x_i(p,m)}{\partial p_j}}_{\text{net effect}} = \underbrace{\left[\frac{\partial h_i(p,u)}{\partial p_j}\right]_{u=v(p,m)}}_{\text{substitution effect}} + \underbrace{\underbrace{-x_j(p,m)}_{\text{wealth lost}}\frac{\partial x_i(p,m)}{\partial m}}_{\text{income effect}}. \tag{10}$$

(vi) * In class, to prove the existence of an equilibrium, we constructed a continuous function and proved that it has a fixed point. Since we only need to consider one price in this economy (why?), this function effectively maps from $\mathbb{R}$ to $\mathbb{R}$. Describe mathematically, and sketch (i.e. draw) this function.

(vii) ** Let (X, d) be any metric space. Prove that if $f, g : X \to \mathbb{R}$ are continuous, then $h(x) = \max\{f(x), g(x)\}$ is also continuous. Hint: you may assume a similar result holds for addition and subtraction.

16: Micro 1, December 2015

Consider a two-generation economy in which both generations consume fish in both time periods. However, the old generation can only work in the first period and the young can only work in the second period. A fishing firm hires workers in each period to catch fish, and a storage firm hires workers to freeze fish in the first time period, and to defrost fish in the second period. Defrosted and fresh fish are perfect substitutes.

(i) Write down a competitive model of the intergenerational fishing economy.

(ii) Is it possible to normalise real wages in the first period to 1?

(iii) Show that if the price of fish in the second period increases, the storage firm sells more fish.

(iv) The government is worried about intergenerational inequality, i.e. that the young will receive lower real wages than the old. It proposes a lump-sum tax on the old and transfer to the young. Show if leisure is a normal good, then this causes at least some prices to change in the new equilibrium.

(v) Suppose it is only possible to store whole fish. Are all equilibria Pareto efficient?

(vi) * Suppose households can home-produce fish storage. Give an example of how this might lead household preferences to be time-inseparable.

(vii) ** Let (X, d_X) and (Y, d_Y) be metric spaces. Prove that if $f : X \to Y$ is continuous and X is compact in (X, d_X), then $f(X)$ is compact in (Y, d_Y).

17: Micro 1, December 2015

Suppose that there are two time periods, and two seasons – summer and winter. There are about ten times as many people in the northern hemisphere than the southern hemisphere. This means that in both periods, an unequal fraction of people experience summer and winter. People prefer to work less and consume more in summer. A firm hires workers to produce a consumption good. It operates in both periods.

(i) Write down a competitive equilibrium model of seasons and hemispheres.

(ii) Suppose the market value of excess demand in all markets in the first time period is positive. Does this mean that there must be excess supply in a market in another time period?

(iii) Using dynamic programming, reformulate the households' problems using net borrowing/lending as a state variable. That is, if this state variable is a positive number for period 1, then the household consumes more than its wages in period 1. The Bellman equation should bury the specifics about consumption or labour decisions in *both* periods.

(iv) Show that households have a decreasing marginal value of net borrowing.

(v) Show that households do more net borrowing (or less net lending) in summer than winter. *Hint: treat "how 'northern' a household is" as a state variable.*

(vi) The United Nations is worried that because of the population imbalance, the seasons create global inequality. They propose achieving equality by requiring everyone to work the same hours during summer and winter. Is it possible to design a lump-sum tax scheme that implements such an allocation? *Hint: assume that leisure is a normal good.*

(vii) ** Prove that the boundary ∂A of any set A is closed.

18: Micro 1, May 2016

Scotland has two major cities, Glasgow and Edinburgh. Suppose that each city has an identical stock of buildings. Workers prefer to consume more buildings, and only benefit from housing located in the city that they choose to work in. There is an electronics factory in each city, that uses labour and buildings to produce electronics. The Glasgow factory is $z > 1$ times as productive as the Edinburgh factory (given the same inputs). To summarise, workers supply labour to factories, consume housing services in their own city, and consume electronics.

(i) Write down a competitive model of the Scottish housing and electronics economy.

(ii) Suppose that there were excess demand for workers and housing in Glasgow, and that the electronics market cleared. Does this imply that there would be excess supply of workers and/or housing in Edinburgh?

(iii) Prove that the Glasgow manufacturer's profit is increasing and convex in its productivity z.

(iv) Prove that if wages in Glasgow increase, then the Glasgow manufacturer demands fewer workers.

(v) Prove that if wages are higher in Glasgow, then rent is also higher in Glasgow.

(vi) Suppose there are several equilibria. Prove that every worker is indifferent between all equilibria.

(vii) * Prove that there is only one equilibrium allocation of resources.

(viii) ** Prove that if f and g are continuous, then $h(x) = f(g(x))$ is continuous.

19: Micro 1, May 2016

According to Seixas, Robins, Attfield and Moulton (1992), coal miners have a 16% risk of developing the disease *black lung*. To keep things simple, suppose that all coal workers must retire early because of their health. Specifically suppose there are two time periods, and workers can choose to work in call centres or coal mines each period. After working in a coal mine, the worker is unable to work thereafter (in any job). However, sick retirees can still enjoy leisure as normal. A firm sells electricity, which it produces with coal miners and call centre workers. Workers supply either kind of labour and consume electricity and leisure.

(i) Write down a competitive model of the electricity markets and the two types of labour markets.

(ii) Reformulate the worker's problem with a Bellman equation, using wealth and health as state variables.

(iii) Prove that in the last period, both professions receive the same wage.

(iv) Prove that the worker has diminishing marginal value of wealth in the last period.

(v) Prove that in the first period, coal miners receive higher wages than call centre workers.

(vi) Suppose the government selects half of the population (e.g. those born in the first half of the year) for a reward, to be funded by lump-sum taxes on the other half of the population. Is this policy Pareto efficient?

(vii) ** Consider the metric space (X, d) where $X = [0, 2]$ and $d(x, y) = |x - y|$. Prove or disprove that $A = [0, 1)$ is an open set.

20: AME, mock exam

Part A. Parts (i), (ii), (iii), and (iv) of Question 19.

Part B.

(i) Let $X = \{(x, y) \in \mathbb{R}^2 : x + y \leq 1\}$. What is the boundary of the set

$$A = \left\{(x, y) \in \mathbb{R}^2_+ : x + y \leq 1\right\}$$

inside the metric space (X, d_2)?

(ii) Consider the sequence of functions $f_n \in CB([0, 1])$ defined by $f_n(x) = x + x/n$. Is f_n a convergent sequence in $(CB[0, 1], d_\infty)$?

(iii) Prove that $(l_\infty([0, 1]), d_\infty)$ is not a compact metric space. (Recall that $l_\infty([0, 1])$ is the set of bounded sequences $x_n \in [0, 1]$.) *Hint: you only need to find one counterexample.*

(iv) Consider any metric space (X, d). Let $x_n, y_n, z_n \in X$ be sequences. Suppose $x_n \to x^*$ and $z_n \to x^*$. Prove that if $d(x_n, y_n) \leq d(x_n, z_n)$ for all n then $y_n \to x^*$.

(v) Write down a recursive Bellman equation for an infinite horizon cake-eating problem in which the size of the cake grows by $r = 0.01 \times 100\%$ every day. Prove that the Bellman operator a contraction on $(CB(\mathbb{R}), d_\infty)$. What is the degree of the contraction? (You do not need to prove that the Bellman operator is a self-map.)

(vi) Let (X, d) be a complete metric space, and $f : X \to X$ be a continuous function. Fix any $x_0 \in X$, and consider the sequence $x_{n+1} = f(x_n)$. Prove that if x_n is a Cauchy sequence then x_n converges to a fixed-point of f.

(vii) Let $X = \{f \in CB([0, 1]) : f(x) = ax \text{ for some } a \in [0, 1]\}$. Is (X, d_∞) a compact metric space?

(viii) Prove that the function

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable at $x^* = 0$.

21: AME, December 2016

Part A

CAF (Construcciones y Auxiliar de Ferrocarriles) produces trams and replacement parts for Edinburgh Trams using labour. Suppose that for each tram used in the first year of operation, 0.2 trams worth of parts must be bought for maintenance before the tram can be used in the second year. Edinburgh Trams produces public transport services from trams and labour to households. Households supply labour to the two companies and consume transport. All households have the same preferences, and shares in all firms are shared equally among all households.

(i) Write down a general equilibrium model of the labour, tram and transportation markets involving households, the factory, and the tram operator over a two-year period. (Hint: Pay careful attention to the depreciation of trams.)

(ii) Write down a Bellman equation for Edinburgh Trams' decision in the first year that buries the second year choices in a value function.

(iii) Show that Edinburgh trams' second year value function is convex in prices.

(iv) Show that if the price of trams increases in the second year, then Edinburgh Trams buys fewer trams in the second year.

Part B

(i) Let (X, d) be a metric space and let $A \subseteq X$. Prove that the boundary of A is a closed set.

(ii) Suppose (X, d) is a compact metric space. Prove that if $A \subseteq X$ is a closed set, then A is a compact set.

(iii) Let (A, d) be a compact metric space. Consider an optimisation problem:

$$\max_{a \in A} u(a),$$

where $u : A \to \mathbb{R}$ is continuous. Prove that the set of optimal choices,

$$A^* = \{a \in A : u(a) \geq u(a') \text{ for all } a' \in A\}$$

is compact. *Hint: use the previous question.*

(iv) Prove that $(CB(\mathbb{R}), d_\infty)$ is *not* a compact metric space. *Hint: you only need one counterexample.*

(v) Suppose that the stock of salmon in the North Sea naturally doubles every five years. Individuals enjoy eating salmon according to a discounted utility function. (a) Write down a recursive Bellman equation to represent the social planner's problem over an infinite time horizon. (b) Sketch a proof that the social value of the stock of salmon is a continuous function. (You do not need to prove that the Bellman operator is a contraction, or prove the principle of optimality.)

(vi) Let $f : X \to X$ be a function on the metric space (X, d). Prove that if f has two fixed points, $x^* \neq x^{**}$, then f is not a contraction.

(vii) Let

$$u(x, y) = \frac{x + y}{1 + y^2 - \sqrt{y}},$$

where $(x, y) \in \mathbb{R}_+ \times [0, 1]$. Find a differentiable lower support function at $x = 2$ for

$$f(x) = \max_{y \in [0,1]} u(x, y).$$

(viii) Suppose that $f : \mathbb{R}_+^{N-1} \to \mathbb{R}$ is strictly concave. Prove that there is at most one solution to the profit maximisation problem,

$$\max_{x \in \mathbb{R}_+^{N-1}} pf(x) - w \cdot x$$

where $(p, w) \in \mathbb{R}_{++}^N$.

22: Micro 1, December 2016

According to the Lincoln Longwool Sheep Breeders Association, the Lincoln Longwool sheep is "one of the most important breeds ever seen in our green and pleasant land." It is a "dual-purpose" breed, meaning it yields high quality wool and meat. Suppose that sheep live for up to two years. If a sheep is killed at the end of the first year, it yields both wool and meat. If a sheep is killed at the end of the second year, it yields wool in both years and the same amount of meat. Households are endowed with sheep, and consume meat and wool each year. Households' preferences can be represented with a discounted utility function. Farms buy sheep to produce wool and meat.

(i) Write down a competitive model of the sheep, wool and meat markets across the two years.

(ii) Prove that farms demand more sheep in the first year if the price of sheep decreases (but no other prices change).

(iii) Write down the firm's value of owning live sheep in the first and second years, making use of a Bellman equation. Prove that these are concave functions of the number of sheep.

(iv) Find an assumption on the model parameters such that the price of sheep decreases over time.

(v) Suppose that half of the population is poor, and only has only half of the sheep endowment. Is it possible to devise a lump-sum transfer scheme that institutes equal welfare for each household?

(vi) * Let $X = \mathbb{R}^6_+$. Suppose there is a continuous function $f : X \to X$ with the properties that (1) $p \in X$ is an equilibrium price vector if and only if $f(p) = p$ and (2) $f(tx) = f(x)$ for all $t > 0$. (a) Apply Brouwer's fixed point theorem to prove that an equilibrium exists. *Hint: you will need to reformulate f.* (b) Fix any $p_0 \in X$. Without using Brouwer's point theorem, prove that if the sequence $p_{n+1} = f(p_n)$ is a Cauchy sequence, then f has a fixed point.

23: Micro 1, December 2016

Suppose a country consists of workers with and without university degrees. Only university graduates can design machines, but both types of worker are equally competent at operating machines. There are two firms: a machine manufacturer that hires university graduates and a clothing manufacturer that buys machines and can hire either type of worker. Workers sell labour and consume clothing and machines (for washing their clothes).

(i) Formulate a competitive equilibrium model of the markets for both types of labour, machines and clothing. *Hint: do not assume that equilibria are symmetric.*

(ii) Suppose at some market prices, the supply of university-educated labour exceeds demand. Does this imply that the demand for uneducated labour exceeds supply?

(iii) Suppose the two firms decide to merge into single firm. (a) Write the combined-firm's profit-maximisation problem using a Bellman equation to separate the output and input choices. (b) Does the equilibrium (or equilibria) change after the merger?

(iv) Prove that if the wages of uneducated workers increases, the clothing manufacturer hires fewer uneducated workers.

(v) Prove that if the clothing manufacturer hires educated workers, then the wages paid to all workers by both firms are equal.

(vi) Suppose every Pareto efficient allocation involves university graduates working for the machine manufacturer only. Is it possible to find lump-sum transfers to implement an allocation in which some university graduates work for the clothing manufacturer?

(vii) * Let $X = \mathbb{R}^4_+$. Suppose there is a continuous function $f : X \to X$ with the properties that (1) $p \in X$ is an equilibrium price vector if and only if $f(p) = p$ and (2) $f(tx) = f(x)$ for all $t > 0$. (a) Apply Brouwer's fixed point theorem to prove that an equilibrium exists. *Hint: you will need to reformulate f.* (b) Fix any $p_0 \in X$. Without using Brouwer's point theorem, prove that if the sequence $p_{n+1} = f(p_n)$ is a Cauchy sequence, then f has a fixed point.

24: AME, May 2017

Part A

Until plastic bottles became popular in the 1960s, milk was sold in glass bottles that could be reused. For simplicity, assume there are two time periods. Suppose households supply labour, and buy bottled milk and empty bottles in both periods. Milk bottles from the first period become empty in the second period, and households can sell these (or buy even more). A firm uses labour to make bottles and bottled milk in both periods.

(i) Write down a competitive model of the bottled milk industry.

(ii) Reformulate the firm's problem by separating the firm's milk and bottle production decisions. *Hint: this is a bit like dynamic programming, but the "Bellman equation" has no choice variables.*

(iii) Prove that the firm has an increasing marginal profit (i.e. a decreasing marginal loss) of a second-period wage increase.

(iv) Prove that the firm reacts to a second-period bottle price increase by increasing its net supply of (empty and filled) bottles.

Part B

(i) Consider the metric space (X, d_2) where $X = [0,1]\times\mathbb{R}$ and $d_2(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$. What is the boundary of the set $A = [0,1] \times \{0\}$ in this space?

(ii) Let $X = \{f : [0,1] \to \mathbb{R}$ s.t. f is continuously differentiable$\}$ and

$$d(f, g) = d_\infty(f, g) + d_\infty(f', g'),$$

where f' and g' are the derivatives of f and g respectively, and $d_\infty(f, g) = \max_{x\in[0,1]} |f(x) - g(x)|$. Prove (a) d is well-defined and (b) (X, d) is a metric space.

(iii) Consider the metric space (X, d_1) where $X = (0,1)$ and $d_1(x, y) = |x - y|$. Supply a counter-example to prove that (X, d_1) is *not* complete.

(iv) Consider the metric space (X, d_1), where $X \subseteq [0,1]$ and $d_1(x, y) = |x - y|$. Suppose that $x_n \in X$ has no convergent subsequence. Prove that X is not a closed set in $(\mathbb{R}, d_1)$.

(v) Let $x_t \in [0,1]$ be the fraction of the population of generation t that is religious. Suppose that each subsequent generation's demographics are deterministic with $x_{t+1} = f(x_t)$, and that $x_t \to x^*$. Prove that if f is a continuous function, then x^* is a fixed point of f, i.e. x^* is a steady state.

(vi) Prove that $f(x) = \frac{1}{3}x^2$ is a contraction on the metric space $(X, d) = ([0,1], d_1)$ where $d_1(x, y) = |x - y|$.

(vii) Consider a two player-game where player one and two choose $a \in [0,1]$ and $b \in [0,1]$ respectively. Suppose that player one and two have best response functions $f(b)$ and $g(a)$ respectively. Let $X = A \times B$ and $h : X \to X$ be defined by $h(a,b) = (f(b), g(a))$. Consider the following procedure (called iterated deletion of dominated strategies) for calculating Nash equilibria:

(a) Set $Y_1 = X$.

(b) Let $Y_{n+1} = h(Y_n)$, that is $Y_{n+1} = \{h(a,b) : (a,b) \in Y_n\}$.

(c) Report $Y_\infty = \cap_{n=1}^{\infty} Y_n$.

Prove that if h is continuous, then $Y_\infty \neq \emptyset$, i.e. that this procedure does not delete all strategies. *Hint: Apply the Cantor intersection theorem.*

(viii) Recall that $CB(\mathbb{R}_+)$ is the set of continuous and bounded functions with domain $\mathbb{R}_+$ and co-domain $\mathbb{R}$, whose distances can be measured with the metric

$$d_\infty(f,g) = \sup_{x \in \mathbb{R}_+} |f(x) - g(x)|.$$

Consider the following Bellman operator $\Phi : CB(\mathbb{R}_+) \to CB(\mathbb{R}_+)$, which is a contraction of degree β on $(CB(\mathbb{R}_+), d_\infty)$:

$$\begin{aligned} \Phi(V)(k) = &\sup_{c,k'} u(c) + \beta V(k') \\ &\text{s.t. } c + k' = g(k). \end{aligned}$$

(You may interpret c as consumption, k as capital, $g(k)$ as output $u(c)$ as utility, and β as the rate of time preference.) Use Banach's fixed point theorem to prove that if u and g are concave, then the fixed point of Φ is concave.

25: Micro 1, May 2017

Consider an economy with two time-periods, in which the entire population lives for both periods. The young and old are identical, except the young have no labour endowment in the first period. They can supply up to their labour endowment and consume food in each period, and have time-separable preferences. A farm produces food from labour.

(i) Devise a competitive model of the food and labour markets.

(ii) Suppose that at the (non-equilibrium) market prices, the market values of the excess demands for food sum to a positive number. Prove that there is excess supply in at least one of the labour markets. *Note: do not assume that there is excess demand in both food markets.*

(iii) Prove that the farm reacts to second-period food-price increases by increasing supply.

(iv) Write down the utility maximisation problem of a "big family" household that makes all market transactions on behalf of the households and the farm. Assume that the big-family household puts equal utility weight on all actual households. *Hints. Recall the home-production example from class. Put the market transactions in one Bellman equation, put the farm choices inside another Bellman equation, and bury the allocation of resources to households inside a value function.*

(v) Suppose the government forcibly reallocated all resources to an efficient egalitarian allocation. If the population were allowed to trade based on their new endowments, what competitive allocation would arise?

(vi) * Give an example of a metric space with the property that every closed subset is compact.

(vii) * Prove the Cantor intersection theorem:

Let (X, d) be a metric space. Suppose $A_n \subseteq X$ is a sequence of sets such that (a) $A_{n+1} \subseteq A_n$, (b) $A_n \neq \emptyset$ and (c) A_n is compact for all n. Let $A = \cap A_n$. Then $A \neq \emptyset$.

26: Micro 1, May 2017

A café and a restaurant both serve meals to customers, using labour and food. The restaurant requires double the labour and food inputs to produce the same number of meals as the café. Households supply labour and only eat at restaurantes and/or cafes. At every level of consumption and supply, households prefer an extra restaurant meal to an extra café meal. A farm produces food from labour only.

(i) Write down a competitive equilibrium model of the labour, food, and meals (restaurants and cafes) markets.

(ii) Suppose there is an equilibrium in which restaurant meals cost £1. Does this mean that there is an equilibrium in which café meals cost £1?

(iii) Prove that in every equilibrium in which café meals are sold, restaurant meals trade at a higher price than café meals.

(iv) Prove that the marginal cost of restaurant meals equals the wage divided by the marginal productivity of labour.

(v) Prove that the restaurant's marginal cost curve is increasing.

(vi) The goverment would like to increase restaurant meal consumption. It proposes (symmetric) lump-sum transfer scheme from households to the restaurant. Would this policy have the desired effect?

(vii) * Give an example of a metric space with the property that every closed subset is compact.

(viii) * Prove the Cantor intersection theorem:

Let (X, d) be a metric space. Suppose $A_n \subseteq X$ is a sequence of sets such that (a) $A_{n+1} \subseteq A_n$, (b) $A_n \neq \emptyset$ and (c) A_n is compact for all n. Let $A = \cap A_n$. Then $A \neq \emptyset$.

27: AME, December 2017

Part A

Internet data centres generate waste energy that can be used to heat homes. Suppose that this waste energy can be transported but not stored. Households benefit more from heat during the evening, and benefit more from the Internet during the day. Households own the data centres, which they rent out to the internet company.

(i) Write down a competitive equilibrium model of the data centre, computing and heat markets during the day and evening.

(ii) Prove that if the daytime heating price increases, then the firm sells more daytime internet services.

(iii) Write down a Bellman equation that separates the household's problem into day and evening choices.

Part B

(i) Let (X, d_X) and (Y, d_Y) be complete metric spaces. Let $Z = X \times Y$ and $d_Z((x, y), (x', y')) = \max\{d_X(x, x'), d_Y(y, y')\}$. Prove that (Z, d_z) is a complete metric space.

(ii) Let (X, d) be any metric space, and $A \subseteq X$ any subset. Provide a counter-example to the following false statement: the interior of the boundary of A is empty, i.e. $\text{int}(\partial A) = \emptyset$.

(iii) Prove that if x is a boundary point of A in (X, d) (defined in terms of sequences), then every open neighbourhood U of x has $U \cap A \neq \emptyset$ and $U \cap (X \backslash A) \neq \emptyset$.

(iv) Prove that $f : X \to Y$ is continuous if and only if for every open ball $U = N_r(y)$, the inverse image $f^{-1}(U)$ is an open set.

(v) Suppose $f : \mathbb{R} \to \mathbb{R}$, where distances in both the domain and co-domain are measured with the Euclidean metric. Suppose that $\lim_{n\to\infty} f(1/n) = 1$ and $f(0) = 0$. Provide an example of an open set U such that $f^{-1}(U)$ is not open.

(vi) Let x_t be the fraction of women that work in professional occupations. Assume that this changes over time according to $x_{t+1} = f(x_t)$ where $f : [0, 1] \to [0, 1]$, and distances are measured by the Euclidean metric. Now, suppose that (i) f is continuous, and that (ii) f is a contraction on $[0, \frac{1}{3})$, and also on $(\frac{1}{3}, \frac{2}{3})$ and $(\frac{2}{3}, 1]$. Prove that there are either two or three steady-states (i.e. fixed points of f).

(vii) Investors with £200000 of assets are able to acquire visas (under some other conditions) to migrate to the United Kingdom. People residing outside the UK receive labour income w each period, and choose how much to consume c and save a', and whether to migrate to the UK. Their utility each period is $u(c)$, which is discounted at rate β. Let $M(a)$ be the value of living in the UK as a migrant with assets a.

Both u and M are bounded and concave. The value of assets to a foreigner $V(a)$ is characterised by the Bellman equation

$$V(a) = \begin{cases} M(a) & \text{if } a \geq 200000 \\ \max_{a'} u(a + w - a') + \beta V(a') & \text{if } a \in [0, 200000). \end{cases}$$

You hope to prove that V is concave with the following strategy – which turns out not to work:

(a) Prove that the Bellman operator is a contraction in $(B(\mathbb{R}_+), d_\infty)$.

(b) Prove that (X, d_∞) is a complete metric space, where

$$X = \{V \in B(\mathbb{R}_+) : V \text{ is concave and } V(a) = M(a) \text{ for } a \geq 200000\}.$$

(c) Prove that the Bellman operator is a self-map on X.

(d) Apply Banach's fixed point theorem.

Which step(s) succeed and which step(s) fail? *You will get credit for checking as many steps as you can.*

(viii) Prove that the following optimization problem (relating to moral hazard) has an optimal solution:

$$\begin{aligned} &\min_{a,b \in \mathbb{R}_+} \; p \exp(a) + (1-p)\exp(b) \\ &\text{s.t. } pa + (1-p)b \geq 1, \text{ and} \\ &a - b \geq q, \end{aligned}$$

where $p \in (0,1)$ and $q > 0$.

28: Micro 1, December 2017

Several identical households enjoy ice cream more in summer than winter, and enjoy soup in winter more than summer. Households are endowed with cows and fishing boats. Ice cream is made from cows. Soup is made from fishing boats. There are only two time periods (winter and summer).

(i) Formulate a competitive equilibrium model of cows, boats, ice cream and soup during summer and winter.

(ii) Suppose that the boat, cow, and icecream markets clear. Does this imply that the soup markets clear?

(iii) Reformulate the households' problem by constructing a value function for both time periods, which are connected via a Bellman equation.

(iv) How does the winter supply of icecream change when the winter price of soup increases?

(v) Is the Pareto frontier of this economy a convex set (under appropriate convexity assumptions about preferences and production)?

(vi) Prove that there is only one competitive equilibrium (under appropriate convexity assumptions about preferences and production).

(vii) * Consider any metric space (X, d), and any two sets A and B with $A \subseteq B \subseteq X$. Prove that if A is open in (X, d), then A is open in (B, d).

(viii) * Let $f : X \to X$ be a function on a complete metric space (X, d). Suppose that $g(x) = f(f(x))$ is a contraction. Prove that f has a unique fixed point.

29: AME, May 2018

Part A

Are improvements in renewable energy technology good news for climate change?

Suppose households own oil deposits, which they can extract and sell at any time. In the first year, only the oil-based power firm operates; it buys oil from households. In the second year, the solar-based power firm is able to hire workers to run solar plants. A pharmaceutical firm hires workers and uses electricity to make medicine. Households only consume electricity and medicine.

(i) Write down a competitive equilibrium model of the power and pharmaceutical industries.

(ii) Prove that the oil-based firm reacts to an energy price decrease in the second year (keeping all other prices fixed) by buying less oil in the second year.

(iii) How does the nature of the solar-based power firm's production function affect the equilibrium amount of oil extracted?

(iv) Decompose the pharmaceutical firm's choices into input and output choices using a Bellman equation.

Part B

(i) What is the interior of $A = \mathbb{Q}^2$ inside the metric space $(\mathbb{R} \times \mathbb{Q}, d_2)$? Recall that $\mathbb{Q}$ is the set of rational numbers, and the Euclidean metric on this space is $d_2(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.

(ii) Consider the metric space $(\ell_\infty(\mathbb{R}), d_\infty)$, i.e. the set of sequences whose absolute values sum to a finite number. Provide an example of a contraction $f : \ell_\infty(\mathbb{R}) \to \ell_\infty(\mathbb{R})$. Recall that $d_\infty(\{x_n\}, \{y_n\}) = \sup_{n \in \mathbb{N}} |x_n - y_n|$.

(iii) Consider any metric space (X, d) and any function $a : X \to \mathbb{R}_{++}$. Let $F_a(X) = \{f : X \to \mathbb{R}, \sup_{x \in X} a(x)f(x) < \infty\}$ and

$$d_a(f, g) = \sup_{x \in X} a(x) d_2(f(x), g(x)).$$

Prove that $(F_a(X), d_a)$ is a metric space. (Boyd, 1990, Journal of Economic Theory used this space to study unbounded value functions.)

(iv) Prove that the metric space $(F_a(X), d_a)$ as defined in the previous question is complete.

(v) Suppose that a person of height $h \in [0, 1]$ has a utility function for food consumption $c \in [0, 2]$ of $u_h(c) = c^h$. Prove that the set of these utility functions, $U = \{u_h : h \in [0, 1]\}$ is a compact subset of the metric space $(CB([0, 2]), d_\infty)$.

Note: you can assume that $f(x, y) = x^y$ and similar functions are continuous. Recall that $CB([0, 2]) = \{f : [0, 2] \to \mathbb{R}, f \text{ is continuous and bounded}\}$, and $d_\infty(f, g) = \sup_{x \in \mathbb{R}_+} |f(x) - g(x)|$.

(vi) Let (X, d) be a non-empty compact metric space, and consider any continuous utility function $u : X \to \mathbb{R}$. Let X^* be the set of optimal choices, i.e. $X^* = \operatorname{argmax}_{x \in X} u(x)$. Prove that X^* is non-empty and compact.

(vii) Consider the optimisation problem:

$$\max_{a,b \in \mathbb{R}_+} p(x - a) + (1 - p)(y - b)$$
$$\text{s.t. } u(a, e) \geq u(b, 0) \text{ and } v(b, 0) \geq v(a, e),$$

where u and v are continuous functions, $p \in [0, 1]$ and $x, y \in \mathbb{R}_+$. Assume that there is a feasible choice $(\bar{a}, \bar{b})$ that satisfies both constraints. Prove that there exists an optimal choice (a^*, b^*).

The economic content of this model – which is not necessary for solving the problem – is as follows. A proportion p of the workers are "good", i.e. they have utility function u, not v, and they produce output x, not y. A recruiter wants to hire an optimal mix of good and bad workers. But he can't tell them apart. Instead, students can put effort e into their education, which the recruiter can observe. So, the recruiter selects wages a for the highly educated and b for lowly educated students.

(viii) Consider the Bellman equation of a firm that makes a profit of $\pi(n)$ when it has $n \in \mathbb{N}$ workers, but it costs $\Delta(n, n')$ to hire (or fire) $n' - n$ workers tomorrow when it has n workers today:

$$V(n) = \sup_{n' \in \mathbb{N}} \pi(n) - \Delta(n, n') + \beta V(n').$$

Assume that π is bounded, $\Delta(n, n) = 0$, $\Delta(n, n') \geq 0$ and $\Delta(n, \cdot)$ is unbounded. Prove that the Bellman equation has exactly one solution.

30: Micro 1, May 2018

Suppose that Idaho farmers each own a field of Russet potatoes, and North Carolina farmers each own a field of sweet potatoes. Assume there are an equal number of farmers in Idaho and North Carolina. All farmers have the same preferences. In this question, do *not* assume that potatoes are (for all prices) normal goods.

(i) Formulate a pure-exchange competitive model of the sweet potato and Russet potato markets.

(ii) Use dynamic programing to reformulate the farmers' utility maximization problem. Specifically, write a Bellman equation that connects the indirect utility function (that gives each farmer's value as a function of prices and endowments) to the expenditure function (that gives each farmer's net expenditure as a function of prices, endowments, and a utility target).

(iii) Suppose that at a particular price level, Idaho farmers respond to a price increase in Russet potatoes by consuming more Russet potatoes. Prove that this implies that Russet potatoes are normal goods for Idaho farmers. *Recall that a good X is an inferior good if the demand for X decreases when the wealth of the consumer increases. Hint 1: apply the envelope theorem to the expenditure function. Hint 2: you may find the Slutsky equation from the lecture notes helpful:*

$$\underbrace{\frac{\partial x_i(p,m)}{\partial p_j}}_{\text{net effect}} = \underbrace{\left[\frac{\partial h_i(p,u)}{\partial p_j}\right]_{u=v(p,m)}}_{\text{substitution effect}} + \underbrace{\underbrace{-x_j(p,m)}_{\text{wealth lost}}\frac{\partial x_i(p,m)}{\partial m}}_{\text{income effect}}.$$

(iv) *For the rest of this question, suppose that your model has two equilibria (while holding all model parameters fixed, including endowments): one in which both types of farmer consume the same things, and one in which North Carolina farmers consume more of both types of potato than Idaho farmers.*

Which of these two equilibria do North Carolina farmers prefer?

(v) Sketch and explain a graph showing a possible shape of the excess demand function of sweet potatoes as a function of the price of sweet potatoes.

(vi) For each of the two equilibria, devise a lump-sum tax policy that implements that equilibrium.

(vii) * Suppose (X, d_X) and (Y, d_Y) are non-empty metric spaces, and consider the metric space $(Z, d_Z) = (X \times Y, d_Z)$ where

$$d_Z(x, y; x', y') = \max\{d_X(x, x'), d_Y(y, y')\}.$$

Prove that if (Z, d_Z) is a compact metric space, then (X, d_X) is a compact metric space.

(viii) * Let (X, d) be a complete metric space, and let $A \subseteq X$. Suppose that $f : X \to X$ is a contraction, and that $f(A) \subseteq A$. Prove that f has a fixed point x^* that lies in the closure of A.

31: AME, December 2018

Part A

There are two car factories, Alfa and Buggy, both of which use machines and labour to make (identical) cars. Assume that Alfa is half as productive, i.e. at the same input levels, it produces half as many cars. Households are endowed with labour and machines, which they rent out to the factories. Households consume cars and leisure.

(i) Write down a competitive equilibrium model of the car, labour and machine markets.

(ii) Suppose Alfa and Buggy merge into a single firm owning the two factories. Write down the merged firm's profit function as a Bellman equation involving the individual firms' profit functions.

(iii) Prove that the merged firm's profit function is convex.

(iv) Prove that Alfa produces fewer cars.

Part B

(i) Consider any metric space (X, d), and any closed ball $A = B_r(x) = \{y \in X : d(x, y) \leq r\}$ with centre $x \in X$ and radius $r > 0$. Prove that A is a closed set.

(ii) Find a counter-example to the following statement: for every set A, the interior of the boundary of A is empty.

(iii) Recall the metric space $(B[0,1], d_\infty)$, where $B[0,1]$ is the set of bounded functions from $[0,1]$ to $\mathbb{R}$ and $d_\infty(f, g) = \sup_{x\in[0,1]} |f(x) - g(x)|$. Consider the set $A = \{f \in B[0,1] : f(x) > 0\}$ and the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x = 0, \\ x & \text{if } x > 0. \end{cases}$$

Prove or disprove that f is a boundary point of A.

(iv) Find a metric d such that $(\mathbb{Q}, d)$ is a complete metric space. Recall that $\mathbb{Q}$ is the set of rational numbers, i.e. ratios of whole numbers.

(v) Prove that $(CB[0,1], d_1)$ is a metric space. Recall that

$$CB[0,1] = \{f : [0,1] \to \mathbb{R},\ f \text{ is continuous and bounded}\}$$

and $d_1(f, g) = \int_0^1 |f(x) - g(x)| dx$.

(vi) Construct an open cover of $(0,1)$ inside the metric space $(\mathbb{R}, d_2)$ that has no finite sub-cover.

(vii) Prove the following version of Banach's fixed point theorem:

Let (X, d) be a compact metric space. If $f : X \to X$ satisfies the property $d(f(x), f(y)) < d(x, y)$ for all $x \neq y$, then f has a unique fixed point.

Hint: $\min_{x\in X} d(x, f(x))$.

(viii) Let $a \in \mathbb{R}_+$ be assets, $e \in \{0,1\}$ be employment status (0 being unemployed), $c \in \mathbb{R}_+$ be consumption, $u(c)$ be the utility of consuming c, w be the wage, $p(0)$ be the probability of finding a job, and $p(1)$ be the probability of keeping a job. Consider the Bellman equation,

$$\begin{aligned} V(a,e) = & \max_{c,a'} u(c) + \beta[p(e)V(a',1) + (1-p(e))V(a',0)] \\ & \text{s.t. } c + a' = a + we. \end{aligned}$$

Suppose that u is concave and bounded.

(a) Define the Bellman operator, including the metric spaces for the domain and co-domain.

(b) Prove that the Bellman operator is a contraction.

(c) Prove that the Bellman equation has a unique bounded solution V^*.

(d) Prove that V^* is strictly concave in a.

32: Micro 1, December 2018

According to EU regulation 543/2011, apples marketed as "class I, colour group A apples" must have "1/2 of total surface red coloured," whereas class II apples have no colour requirements.

Suppose that a farm make class I and class II apples out of labour. The farm can not control what fraction of apples are of each class, only the total number of apples grown. A beverage firm makes apple juice out of labour and apples – the two classes of apple are perfect substitutes as far as the firm is concerned. Households sell labour and buy both types of apples and apple juice; they prefer class I apples over class II apples.

(i) Formulate a competitive model of the labour, apple, and apple juice markets.

(ii) Prove that if the beverage firm buys both types of apples, then the two types of apples trade at the same price.

(iii) Suppose the farm and the beverage firm merge into a single firm. Formulate the merged firm's profit function.

(iv) Suppose at some equilibrium, the merged firm uses some class I apples for apple juice production. Prove that if the price of class I apples decreases (to a non-equilibrium price), then the merged firm responds by allocating more apples to apple juice production.

(v) What effect on prices and quantities would a lump-sum transfer from the beverage company to the farm have?

(vi) * Provide a counter-example to the following false conjecture: $(B(\mathbb{N}, [0, 1]), d_\infty)$ is a compact metric space, where $B(\mathbb{N}, [0, 1])$ is the set of bounded functions from the natural numbers to $[0, 1]$, and $d_\infty(f, g) = \sup_{n\in\mathbb{N}} |f(n) - g(n)|$.

(vii) * Prove the following generalisation of Cantor's intersection theorem:

Let (X, d) be a complete metric space. Define the diameter of a set $A \subseteq X$ as $\text{diam}(A) = \sup_{a,b\in A} d(a, b)$. Let $A_n \subseteq X$ be a sequence of non-empty closed sets. If $A_{n+1} \subseteq A_n$ and $\text{diam}(A_n) \to 0$ then $\cap_{n=1}^{\infty} A_n$ contains a single point.

33: Micro 1, May 2019

A cosmetics firm makes perfume using consulting services. Consulting services are produced from specialised labour and lab materials. There are two consultants, who each owns and supplies labour exclusively to his own consulting firm. The consultants are endowed with the same amount lab materials, which they can sell to any firm. The old consultant has double the human capital of the young consultant. The two consultants own an equal share in the cosmetics firm, and consume perfume only (but not leisure or lab materials).

(i) Formulate a competitive model of the perfume, consulting, labour, and lab material markets. Hint: you might find it easier to model labour markets as rental markets for human capital.

(ii) Suppose that at some non-equilibrium price vector, all markets clear except the labour markets. Does this mean that one of the labour markets has excess supply?

(iii) Prove that if the price of lab materials increases, then the old consulting firm responds by purchasing fewer lab materials.

(iv) Write down the value function for the old consulting firm after it has chosen the labour demand but before it has chosen the demand for lab materials. Write down a Bellman equation linking this to the profit function.

(v) Prove that if the consulting production function has constant returns to scale, then the old consultant uses more lab materials than the young consultant. Hint: differentiating $f(tx) = tf(x)$ with respect to x_1 gives $tf_1(tx) = tf_1(x)$.

(vi) The government would like to increase the amount of perfume production. Either devise a lump-sum transfer scheme that would increase perfume production, or prove that this is impossible.

(vii) * Consider the function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$, the metric space $(X, d) = (C([0,1]), d_\infty)$, and the set $A = \{f \in X : f(0) \geq 0\}$, where

$$C([0,1]) = \{f : [0,1] \to \mathbb{R}, f \text{ is continuous}\}$$
$$d_\infty(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|.$$

Is f in the interior of A?

34: AME, May 2019

Part A.

Households are endowed with time and vegetables, which they can consume or sell to a restaurant. The restaurant produces meals, which households can also consume.

(i) Formulate a competitive model of the labour, vegetable and meal markets.

(ii) Suppose the manager of the restaurant has already decided how many vegetables to buy, but the chef still needs to decide how many cooks to hire. Write down the chef's value function. Write down a Bellman equation that connects the chef's value function to the restaurant's profit function.

(iii) Prove that the chef's value function is convex in prices.

(iv) Prove that the chef's demand for cooks is decreasing in the wage of cooks.

Part B.

(i) Find a metric space (X, d) such that $x_n = \frac{1}{n}$ is not a Cauchy sequence.

(ii) Find a counterexample to the following false statement: Let (X, d) be a metric space. If A is closed and bounded inside (X, d), then A is compact.

(iii) Prove that the optimisation problem

$$\max_{x \in [100,101], y \in \{0,1\}} (x + y) \sin x$$

has an optimal solution (x^*, y^*).

(iv) Let (X, d) be a metric space. Let $\mathcal{U}$ be a set of open sets. Prove or disprove that the union of these sets, $A = \cup \mathcal{U}$, is an open set. Note: $\mathcal{U}$ might be an infinite set.

(v) Consider the metric spaces (X, d) and (X, d') where $d'(x, y) = \min \{1, d(x, y)\}$. Prove that if (X, d) is complete, then (X, d') is complete.

(vi) Consider the metric spaces (X, d_X), (Y, d_Y) and $(CB(X, Y), d_\infty)$, where

$$CB(X, Y) = \{f : X \to Y, f \text{ is continuous and bounded}\}$$
$$\text{and } d_\infty(f, g) = \sup_{x \in X} d_Y(f(x), g(x)).$$

Prove that if $x_n \in X$ and $f_n \in CB(X, Y)$ are convergent with $x_n \to x^*$ and $f_n \to f^*$, then $y_n = f_n(x_n)$ is convergent with $y_n \to y^* = f^*(x^*)$.

(vii) Let (X, d) be a compact metric space. Consider the metric space $(C(X), d_\infty)$ of continuous functions $C(X) = \{f : X \to \mathbb{R},\ f \text{ is continuous}\}$ and

$$d_\infty(f, g) = \sup_{x \in X} d(f(x), g(x)).$$

Consider the function $T : C(X) \to X$ defined by $T(f) = \max_{x \in X} f(x)$. Prove that T is well-defined.

(viii) Let (X, d) be a complete metric space, and let f_n be a sequence of contractions on (X, d) of degree a. Prove that there exists a unique solution x_n^* to the system of equations $x_n = f_n(x_{n+1})$.

35: AME, December 2019

Part A

During the early industrial revolution, trains carried grain from the American midwest to the east coast, and clothing from the east coast to the midwest. Train transport capacity was produced by labour. Capacity was bi-directional, in the sense that the total capacity required was the maximum of the east-bound and west-bound capacity used. Train companies were vertically integrated with retail, i.e. train companies bought and sold grain and clothes. Farms made grain from labour, and factories made clothes from labour. Households in both locations supplied labour and consumed grain and clothes. Train transport could utilise labour from either location, whereas farms only used midwest labour, and factories only used east coast labour. All households had the same labour endowment, and owned the same shares in all firms.

(i) Formulate a competitive model of the labour, grain and clothing markets in both locations.

(ii) Prove that the train firm's profit function is convex.

(iii) Prove that the train firm reacts to an increase in the midwest price of grain by trading less grain.

(iv) The train companies ("robber barons") often purchased upstream suppliers. Write down the profit function of the train firm after it purchased the other firms as a Bellman equation.

Part B

(i) Consider the metric space (X, d_1) where $X = [0, 2] \times \{0, 1\}$ and $d_1(x, y) = |x_1 - y_1| + |x_2 - y_2|$. What is the interior of the set $A = [0, 1] \times \{0\}$ inside (X, d_1)?

(ii) Suppose A is a subset inside the metric space (X, d). Prove that if A is both closed and open, then the boundary of A is empty, i.e. $\partial A = \emptyset$.

(iii) Let (X, d) be a metric space. Prove that if $A \subseteq X$ and (A, d) is a complete metric space, then A is a closed set inside (X, d).

(iv) Give an example of a complete and bounded metric space that is not compact.

(v) Prove that if X is a finite set, then the metric space (X, d) is compact.

(vi) Consider a public good contributions game in which player 1 donates x and player 2 donates y. Suppose player 1 wants to donate $x = f(y)$ and player 2 wants to donate $y = g(x)$, where f and g are decreasing differentiable functions with $f'(y) > -a$ for all x and $g'(x) > -a$ for all y, and a is some number in $(0, 1)$. Prove that there is a unique equilibrium, i.e. (x^*, y^*) such that $x^* = f(y^*)$ and $y^* = g(x^*)$.

(vii) Consider a social planner who would like to distribute an endowment $e > 0$ among a society of n individuals to maximise welfare $W(x) = \sum_{i=1}^{n} u_i(x_i)$, where each individual's utility function $u_i : \mathbb{R}_+ \to \mathbb{R}$ is continuous. Prove that there is a solution to the social planner's problem,

$$\max_{x \in \mathbb{R}^n_+} W(x)$$
$$\text{s.t.} \sum_{i=1}^{n} x_i = e.$$

(viii) Suppose a boiler's energy efficiency degrades over time, but can be restored. Let x be the boiler's efficiency – measured in the amount of energy needed to heat a building for one day, p be the price of energy, $r(x, x')$ be the repair cost to restore x to x' (which might be greater than zero even if $x' > x$, due to degradation). Assume that r is continuous. Money is discounted at rate β. The value of boilers $\pi(x)$ solves the Bellman equation

$$\pi(x) = \inf_{x' \in [0,1]} px + r(x, x') + \beta\pi(x').$$

Recall: $\inf A$ *is the largest number that is weakly smaller than everything in* A*, e.g.* $\inf (0, 1] = 0$.

(a) Reformulate the Bellman equation as a fixed point problem.

(b) Assume that the function in the previous part is a contraction. Suppose that $r(x, x')$ is concave in x. Prove that the solution to the Bellman equation, π, is concave. *Hint: this proof has several steps. As always, you can get credit for any of the steps.*

36: Micro 1, December 2019

After the Forth bridge was built in 1889, trade and commuting between Fife and Edinburgh became much easier.

Coal is produced from labour. Garments (clothes) are produced from coal and labour. Assume that both coal and garments can be produced in both places, but that coal is easier to produce in Fife, and garments are easier to produce in Edinburgh. Before the bridge was completed, Edinburgh and Fife were autonomous, i.e. there was no trade between them. Afterwards, workers from both places could commute and work in either place, and coal and garments were traded. Assume that households have discounted utility, with the same per-period utility function. Assume that all households are identical in terms of endowments and preferences apart from (i) their locations, and (ii) that all firms are owned locally. Neither coal nor garments are storable.

(i) Formulate a competitive model of the labour, coal and garments markets operating before and after the Forth bridge was completed. *Hint: any correct answer has more than seven markets.*

(ii) Re-formulate the Edinburgh households' utility maximisation problems by burying the post-bridge-opening choices inside a value function.

(iii) Prove that in every equilibrium, Edinburgh households neither save nor borrow (where dividends from profits earned in each period are attributed to that period).

(iv) Suppose that when the bridges open, the real wages in Edinburgh in terms of coal increases, i.e. the ratio of wages divided by the price of coal in Edinburgh increases. Prove that Edinburgh decreases its coal production.

(v) Prove that in every equilibrium, welfare in Edinburgh increases after the bridge opens.

(vi) Prove that in every equilibrium, after the bridge opens, Edinburgh produces more garments than Fife.

37: AME, May 2020

Part A

A machine learning (ML) firm has to choose between Canada and Germany for its next data centre. The ML firm uses energy and labour to provide ML services. Both energy and labour must be supplied locally (for energy efficiency and security reasons). A car firm uses energy, ML services and labour – all from any location – to make electric cars. Households are endowed with energy and labour which they supply to the market, and they consume cars. There is a single global market for cars and ML, but separate markets in each country for labour and energy.

(i) Formulate a competitive model of the energy, labour, ML services, and electric car markets in Canada and Germany.

(ii) Suppose the ML firm chooses Canada. Prove that the ML firm reacts to a Canadian wage rise by hiring fewer Canadian workers.

(iii) Formulate the ML firm's problem as a Bellman equation involving the location choice only.

Part B

(i) Either give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ with the property that $f([0,1)) = [0,1]$, or prove that this is impossible.

(ii) Either give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ with the property that $f([0,1]) = [0,1)$, or prove that this is impossible.

(iii) Consider the metric spaces (X, d_X), (Y, d_Y), (Z, d_Z), and $(X \times Y, d_\infty)$ where

$$d_\infty(x, y; x', y') = \max\{d_X(x, x'), d_Y(y, y')\}.$$

Suppose that $f : X \times Y \to Z$ is continuous. Prove that if $(x_n, y_n) \to (x^*, y^*)$ inside $(X \times Y, d_\infty)$ then

$$\lim_{m\to\infty} \lim_{n\to\infty} f(x_m, y_n) = f(x^*, y^*).$$

(iv) Let (X, d), and (X, d') be two metric spaces. Suppose that both metric spaces have the same open sets, i.e. U is open inside (X, d) if and only if U is open inside (X, d'). Consider any sequence $x_n \in X$. Prove that x_n is convergent inside (X, d) if and only if it is convergent inside (X, d').

(v) Let $X = [-1, 1]$. Recall that $(B(X), d_\infty)$ is the metric space defined by

$$B(X) = \{f : X \to \mathbb{R} \text{ s.t. } f \text{ is bounded}\}$$

and $d_\infty(f, g) = \sup_{x\in X} |f(x) - g(x)|$. We say that a function $g \in B(X)$ is a polynomial if there exist numbers $a_0, a_1, \cdots, a_n \in \mathbb{R}$ such that $g(x) = \sum_{i=0} a_i x^i$ for all $x \in X$. Let $P(X)$ be the set of all such polynomials. Consider the function $f^* \in B(X)$ defined by

$$f^*(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Prove that there is no sequence $f_n \in P(X)$ such that $f_n \to f^*$ inside $(B(X), d_\infty)$. *Hint: polynomials are continuous, and the continuous bounded functions form a complete metric space.*

(vi) Consider the compact metric spaces (X, d_X), (Y, d_Y), and $(X \times Y, d_\infty)$ where

$$d_\infty(x, y; x', y') = \max \{d_X(x, x'), d_Y(y, y')\}.$$

Consider a function $f : X \times Y \to X \times Y$. Suppose that for all $x \in X$, the function $g(y) = f_2(x, y)$ is a contraction on (Y, d_Y), and similarly $f_1(\cdot, y)$ is a contraction on (X, d_X) for all $y \in Y$. Prove that f has a fixed point in $(X \times Y, d_\infty)$.

(vii) Suppose a country is either in a boom ($x = 1$) or recession ($x = 0$), which affects its tax revenue of t_x. Recessions occur each period with probability p. It has savings of a. It can use a and t_x to finance government spending g and future savings a' at interest rate $1/r - 1$. Government programmes g give a utility $u(g)$ each period which is discounted by β, where u is increasing, continuous, concave and bounded. Its Bellman equation is

$$\begin{aligned} V(x, a) = \sup_{g, a' \geq 0} \; & u(g) + \beta[pV(0, a') + (1 - p)V(1, a')] \\ \text{s.t.} \; & g + ra' = t_x + a. \end{aligned}$$

(a) Formulate an appropriate domain of the corresponding Bellman operator. *Hint: ensure it is a complete metric space.*

(b) Prove that the Bellman operator is a contraction.

(viii) Suppose a hospital uses n nurses and v ventilators to treat Coronavirus infections, which trade a prices w and p respectively. Both nurses and respirators are needed to save lives. The hospital saves $f(n, v)$ lives, whom it values at x each. The hospital solves the following problem:

$$\sup_{n,v} xf(n, v) - wn - pv.$$

Assume that f is continuous and has constant returns to scale, i.e. $f(an, av) = af(n, v)$ for all (n, v) and all $a > 0$. Prove that there is an optimal v/n ratio.

38: Micro 1, May 2020

Consider an economy consisting of two types of firms – startups and farms, and two types of household – engineer and unskilled. A startup produces a completely new type of vegan snack food. It has two possible ways to operate. First, it could use engineers and vegetables only. Alternatively, it could use 5000 engineer-hours to develop a more efficient constant-returns-to-scale technology that transforms engineering labour, unskilled labour, and vegetables into vegan snack food. In the second case, engineers have a higher marginal product than unskilled workers. A farm hires workers to make vegetables (either type is equally productive). Some households are endowed with engineering labour, and the other households are endowed with unskilled labour. Households sell their labour inelastically, and consume both vegetables and vegan snack food.

(i) Formulate a competitive model of the vegan snack food, vegetables, engineering labour, and unskilled labour markets. Formulate the startup's problem using a Bellman equation involving a choice between the two alternatives.

(ii) Is it possible for there to be excess supply of both types of food and both types of labour (at non-equilibrium prices)?

(iii) Prove that in every equilibrium, unskilled workers receive a smaller or equal wage than the engineers.

(iv) Suppose the startup develops the more efficient technology. Prove that the startup responds to an engineering wage increase by hiring fewer engineers.

(v) Suppose that there is only one equilibrium, and it involes the startup pursuing the first alternative. The government would like to increase entrepreneurial activity, i.e. to ensure that the start-up pursues the second alternative. When is it possible to design a lump-sum tax scheme to do this?

(vi) * Formulate the excess demand function of the economy, and use it to express the market-clearing conditions.

39: AME, December 2020

Part A

Suppose all households are endowed with broken mobile phones. Half of the households have new model broken phones, and half have old model broken phones. They sell their broken phones. Repair shops hire workers and buy broken phones, and sell working phones (both models). Factories hire workers and make (working) new model phones. Households choose how many phones of each type to buy, and how much labour to supply.

(i) Write down a competitive model of the four phone markets (broken/working, new/old model) and the labour market.

(ii) Recall $f : \mathbb{R}^n \to \mathbb{R}$ is strictly concave if for all $t \in (0, 1)$, and all $x, x' \in \mathbb{R}^n$,

$$tf(x) + (1-t)f(x') < f(tx + (1-t)x').$$

Suppose there are 10 repair shops, and their production function is strictly concave. Prove that all 10 repair shops repair the same number of phones.

(iii) Suppose the phone manufacturer buys all of the repair shops. Write down the conglomerate's profit function using a Bellman equation.

(iv) Calculate the marginal repair shop profit of a wage increase.

Part B

(i) Let $f_n(x) = \frac{x}{n}$ and $A = \{f_n : n \in \mathbb{N}\}$. Is A a compact set inside $(B[0, 1], d_\infty)$?

(ii) Let $F = B((0, 1))$ be the set of bounded functions with domain $(0, 1)$ and co-domain $\mathbb{R}$. Consider the sequence of functions $f_n \in F$ defined by $f_n(x) = \frac{x}{n}$. Find a metric d such that (F, d) is a metric space and f_n is not convergent.

(iii) Find a counter-example to this false conjecture. Suppose (X, d_X) and (Y, d_Y) are metric spaces. Consider the metric space (Z, d_Z), where $Z = X \times Y$ and

$$d_Z(x, y; x', y') = d_X(x, x') + d_Y(y, y').$$

If A is a closed set inside (Z, d_Z), then $A_X = \{x : (x, y) \in A\}$ is a closed set in (X, d_X).

(iv) Consider the metric spaces (X, d_X), (Y, d_Y) and (Z, d_Z) and sets A and A_X defined in the previous question. Prove that if A is a compact set inside (Z, d_Z), then A_X is a compact set inside (X, d_X).

(v) Let (X, d) be a compact metric space, A be a closed set, B be an open set with $A \subseteq B$. Prove that there exists a continuous function $f : X \to \mathbb{R}$ such that (i) $f(x) = 0$ for all $x \in A$, and (ii) $f(x) > 1$ for all $x \notin B$. *Hint: You may make use of the following theorem without proving it: d is continuous.*

(vi) Consider the set of wealth distributions (Lorenz curves),

$$X = \{f \in C(\mathbb{R}, [0,1]) : f \text{ is a weakly increasing}\}.$$

Prove that (X, d_∞) is a complete metric space.

(vii) Consider the set of wealth distributions X from the previous question. Suppose that today's wealth distribution is $f_0 \in X$. Margaret Thatcher's poll tax transforms year n's distribution, f_n, into $f_{n+1} = T(f_n)$ the following year, where T is a contraction. Robin Hood does not like Margaret Thatcher's T function, so he tries to undo it by replacing f_0 with $\hat{f}_0$. Margaret Thatcher's T function applies thereafter, i.e. $\hat{f}_{n+1} = T(\hat{f}_n)$. Explain why Robin Hood's intervention is ineffective in the long run.

(viii) Let $a \in [0,1]$ be the quality of a factory, $r \in [1.01, 2]$ be the interest rate, w be wages, $h \in [0,1]$ be hours of work. Every period, the factory has to decide whether to shutdown (permanently), and how much work to put into maintenance of the factory. Its Bellman equations are

$$V(a,r) = \frac{1}{r} \max\{0, W(a,r)\}$$
$$W(a,r) = \max_{h \in [0,1]} a - wh + V(\sqrt{ah}, r).$$

(a) Reformulate the factory's problem using a single Bellman equation with V on both sides.

(b) What is the corresponding Bellman operator? Don't forget to specify the metric space for the domain and co-domain.

(c) Assume that the domain is a complete metric space, and the operator is a contraction. Prove that V is strictly decreasing in r.

40: Micro 1, December 2020

Suppose that everyone knows how to make pizza. A household of Russian Jews moves to Edinburgh, and opens a restaurant selling blintzes (a sweet crepe), and a school teaching the locals how to cook blintzes. The course is indivisible, takes a year to complete, and runs in the first year only. The households allocate their time in two years between leisure, studying, and supplying unskilled or skilled (in blintzes) labour. Each household chooses how much of each type of food to eat each year. The Russian household owns the Russian restaurant/school, and the other households hold equal shares in the pizza restaurant. Assume that all households do not value leisure, and have undiscounted utility functions, i.e. they are perfectly patient.

(i) Formulate a competitive model of the labour, food and education markets over two years.

(ii) Prove that if the first-period skilled wages increase, then the Blintzes firm trains fewer people in the first period.

(iii) Suppose that in equilibrium, at least one household studies blintzes. Prove that more blintzes are produced in the second period than the first.

(iv) Reformulate the local households' problem into a dynamic programming problem in which labour and education are chosen first, and consumption choices are buried inside a value function.

(v) Prove that in every equilibrium in which both pizza and blintzes are consumed, the local households eat the same food as each other (regardless of the amount of study they do). Hint: assume the utility function is strictly concave.

(vi) The locals notice that the Russian's firm is making a huge profit. They want to restore perfect equality to Edinburgh. They propose a 100% tax on all firms' profits, and distributing these equally among the households. Would this deliver an efficient and equal equilibrium?

41: AME, May 2021

Part A

Suppose that half of the houses in a city are in polluted areas. Residents in polluted areas suffer health problems, and can only do unskilled work. Apart from this problem, all workers can do skilled and unskilled work. A firm hires skilled and unskilled workers to make furniture. Workers are endowed with a house and hours which they can sell. Workers can buy houses and furniture. Workers can not live together (due to fire regulations).

(i) Formulate a competitive model of the housing, labour and furniture markets.

(ii) Reformulate the worker's problem using a Bellman equation with a a housing choice, and with the other choices buried inside a value function.

(iii) Write down a formula for the marginal profit of a skilled wage increase.

(iv) Suppose that the firm's production function is strictly concave. Prove that the firm has at most one optimal choice.

Part B

(i) (easy) Let $f : X \to Y$ be a continuous function between two metric spaces (X, d_X) and (Y, d_Y). Prove or disprove that $f(\partial A) = \partial f(A)$ for all sets $A \subseteq X$. *Note:* ∂A *denotes the set of boundary points of* A*, and* $f(A) = \{f(a) : a \in A\}$.

(ii) (easy) Consider the metric space $(\mathbb{R}, d)$, where d is the discrete metric. Find a contraction $f : \mathbb{R} \to \mathbb{R}$ on this space.

(iii) (easy) Consider the sequence $f_n : \mathbb{N} \to [0, 1]$ defined by

$$f_n(x) = \begin{cases} 1 & \text{if } x < n, \\ 0 & \text{if } x \geq n. \end{cases}$$

Prove that this sequence is not convergent in $(B(\mathbb{N}), d_\infty)$.

(iv) (medium) Find a metric d such that the sequence f_n in the previous question converges to $f^*(x) = 1$.

(v) (medium) Let $X = [0, 1)$. Find a metric d such that (X, d) is a compact metric space.

(vi) (medium) Suppose you are considering buying a house at market price p, which you value at v. But you don't want to buy if it has any (major) defects. You have taken a quick look already, and you think the probability of defects is q. You can pay inspectors c for conditionally independent reports about the house, which have type 1 and 2 errors of x and y. Each day, you choose whether to buy the house, to buy another report, or to give up. You discount days at rate β. You have a Bellman equation

$$\begin{aligned} V(q) = \max\{0, qv - p, \\ & - c + [qx + (1-q)y]\beta V\left(\frac{qx}{qx + (1-q)y}\right) \\ & + [q(1-x) + (1-q)(1-y)]\beta V\left(\frac{q(1-x)}{q(1-x) + (1-q)(1-y)}\right)\right\}. \end{aligned}$$

Prove that the optimal policy involves giving up for low q, i.e. there exists $q_1 \in [0,1]$ such that giving up is optimal for all $q \in [0, q_1]$.

(vii) (hard) Define

$$f^n(x) = \begin{cases} f(f^{n-1}(x)) & \text{if } n \geq 1 \\ x & \text{if } n = 0. \end{cases}$$

Prove or disprove: if $f : [0,1] \to [0,1]$, $f(1) = 1$, and $f(x) > x$ for all $x < 1$, then $\lim_{n\to\infty} f^n(0) = 1$.

(viii) (hard) Suppose (X, d_X) and (Y, d_Y) are metric spaces, where $K = X \cap Y$ is a compact set in both spaces. Suppose that $d_X(a,b) = d_Y(a,b)$ for all $a, b \in K$. Let $Z = X \cup Y$. Construct a metric d_Z on Z such that $d_Z(a,b) = d_X(a,b)$ for all $a, b \in X$, and $d_Z(a,b) = d_Y(a,b)$ for all $a, b \in Y$. *Hint: use the fact that d_X and d_Y are continuous. Note: a complete proof is long, with lots of cases to consider. You can get an almost perfect score for the "proofs" learning outcome by showing 1 or 2 cases well.*

42: Micro 1, May 2021

Write down a two-period model in which a pandemic strikes in the second period. In both periods, all households split their time between working, studying music online and/or offline (for leisure), and other leisure (e.g. watching free videos). A music company hires workers to supply piano lessons, and a restaurant hires workers to make meals. In the second period, two things change. First, music lessons move online, which is less fun for the student, and more work for the teacher. Second, people must eat restaurant meals at home, which is less fun. All workers can do both jobs. All households own an equal share of the firms.

(i) Formulate a competitive model of the music education, labour, and restaurant meal markets.

(ii) Reformulate the households' problem using a Bellman equation in which the second period choices are buried inside a value function.

(iii) Assume that the relevant utility functions are strictly concave. Prove that in the Bellman equation you just wrote down, there is at most one optimal savings choice.

(iv) The government would like to encourage more people to work in the second period. To this end, it plans a lump sum tax on workers, which funds a subsidy to the firms in the second period. Would this policy lead to more work in the second period?

(v) Suppose that at market prices, all labour markets clear, all education markets clear, and that all households save none of their money for the second period. Prove that this implies that both restaurant meal markets clear. *Hint: think carefully about what it means to save for the future.*

43: AME, December 2021

Part A

Suppose there are three years only. A fish farmer owns a fish farm, and is endowed with some adult trout (a fresh water fish). Each adult trout on the farm has children, which take one year to mature into adults. However, the children need to be cared for, otherwise many of them will die. Therefore, the fish farm hires workers to increase the fraction of child trout that survive. All households – both the farmer and the workers – choose how much labour to supply in years one and two, and how much trout to eat in all three years. A farm buys adult trout and hires workers in one year to make adult trout the following year.

(i) Formulate a competitive model of the two labour markets and three fish markets.

(ii) Reformulate the farm's problem using a Bellman equation, in which the second and third year choices are buried inside a value function.

(iii) Prove that the firm's profit function is convex in the price of fish in the first year.

(iv) What is the derivative of the firm's profit function with respect to the price of fish in the second year?

Part B

(i) (Easy) Let (X, d) be a metric space. Find a counter-example to the false hypothesis, that every open ball $B_r(x)$ is connected.

(ii) (Easy) Let (X, d) be a metric space. Prove that if every set $A \subseteq X$ is open, then every set $A \subseteq X$ is closed.

(iii) (Easy) Consider the metric spaces (X, d_X) and (Y, d_Y). Suppose the function $f : X \to Y$ is bijective, and that f and f^{-1} are continuous. Prove that if $g : X \to X$ is discontinuous, then $h : X \to Y$ defined by $h(x) = f(g(x))$ is discontinuous.

(iv) (Easy) A beer monopolist spends $c(q)$ to make q units of beer. He offers different prices to students and workers of p_s and p_w, respectively. Students and workers have inverse demand curves, $q_s, q_w : \mathbb{R}_+ \to \mathbb{R}_+$, which are continuous, decreasing and $q_s(\bar{p}) = q_w(\bar{p}) = 0$ for some price $\bar{p} > 0$. Also assume that $c : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous. The monopolist's problem is

$$\begin{aligned} &\max_{p_s, p_w} p_s q_s(p_s) + p_w q_w(p_w) - c(q_s(p_s) + q_w(p_w)) \\ &\text{s.t. } p_s \geq 0 \text{ and } p_w \geq 0. \end{aligned}$$

Prove that there is an optimal solution, (p_s^*, p_w^*) to the monopolist's problem.

(v) (Medium) Consider the metric space (X, d) where $X = \mathbb{R}_+ \cup \{\infty\}$ and

$$d(x, y) = \begin{cases} \min\{1, |x - y|\} & \text{if } x, y \in \mathbb{R}_+, \\ 0 & \text{if } x = y = \infty, \\ 1 & \text{otherwise.} \end{cases}$$

Prove that (X, d) is not compact.

(vi) (Medium) Find a counter-example to the following false claim: If (X, d) is a complete metric space and $f : X \to X$ is a contraction, then $f(X)$ is connected.

(vii) (Medium) Prove that (X, d) is connected if and only if every continuous function $f : X \to \{0, 1\}$ is constant.

(viii) (Hard) A retired woman wakes up with a bank balance b and cash c in her wallet. She chooses how much of her cash to spend each day x, and how much leisure time to have, $\ell \leq 24$, which gives her utility $u(x, \ell)$. If she wants more cash, she has to walk 2 hours (round trip) to the bank. She doesn't withdraw all of her bank balance, because cash is exposed to inflation i, and banks pay interest on deposits to cancel out inflation. She discounts the future at rate β. Her value function solves the Bellman equation

$$\begin{aligned} V(b, c) = & \sup_{x, b', c' \geq 0} u(x, 24 - 2I(b' \neq b)) + \beta V(b', c') \\ & \text{s.t. } x + b' + c'(1 + i) = b + c, \end{aligned}$$

where $I(\cdot)$ is the indicator function, i.e. $I(b' \neq b)$ equals 1 if $b' \neq b$ and 0 otherwise. Suppose the utility function u is unbounded. Specify a suitable metric space for the domain of the Bellman operator, such that Banach's fixed point theorem can be applied to the Bellman operator. *Hint: divide the value functions by the maximum utility that can be achieved in one day.*

44: Micro 1, December 2021

Consider the economy of two nearby and identical towns, Byron Bay and Casino. They are near enough to share a hospital, and households are indifferent between travelling to a hospital in either town. However, the towns are too far apart for workers to commute. A hospital requires many workers before it treats its first patient. Therefore, assume it is inefficient for both towns to operate their own hospital. On the other hand, both towns have a resort, which are perfect substitutes. Workers in each town supply labour to one of the local businesses, and consume holidays and treatments. Workers own equal shares in the local businesses. Hospitals and resorts only use labour to supply treatments and holidays.

(i) Construct a competitive model of Byron Bay and Casino. *Hint: assume that there are two hospitals, but accommodate the possibility that the hospitals are inactive, i.e. hire no workers.*

(ii) Suppose the resorts merge into a single firm. Write down the merged firm's profit function, with and without a Bellman equation.

(iii) Is there a lump-sum transfer scheme that implements perfect equality?

(iv) Suppose that all markets clear except the holiday markets in the two cities. Does this imply that both holiday markets clear? Does your answer depend on whether you model these two markets as a single market? Explain.

(v) Consider an equilibrium in which a hospital opens in Byron Bay only. Prove that the residents of Byron Bay work more than those of Casino.

45: AME, May 2022

Part A

A computer processor is faster if it has fewer defects, because the defective components must be disabled. For example, it might have fewer arithmetic units or less cache memory.

A processor factory ("fab") hires workers for two tasks: production and quality control. The fab sells two types of processor: fully functional (fast) and defective (slow). Workers choose what processors to buy, allocate their time between work and leisure, own an equal share of the fab, and are all identical.

(i) Formulate a competitive model of the labour market and the two processor markets.

(ii) Reformulate the fab's profit maximisation problem using a Bellman equation in which the firm's choice of how to allocate its labour force across the two tasks is buried inside a value function.

(iii) Prove that if wages increase, then the fab hires fewer workers.

Part B

(i) (Easy) Recall that $CB(\mathbb{R})$ is the set of continuous and bounded functions whose domain and co-domain is $(\mathbb{R}, d_2)$. Let $X = \{f \in CB(\mathbb{R}) : f(0) = 0 \text{ and } f(x) \geq 0\}$. Prove that (X, d_∞) is a complete metric space.

(ii) (Easy) Let (X, d_X) and (Y, d_Y) be metric space. Consider (Z, d_Z), where $Z = X \times Y$ and $d_Z(x, y; x', y') = d_X(x, x') + d_Y(y, y')$. Prove that if U is an open set inside (Z, d_Z), then $V = \{x \in X : (x, y) \in U\}$ is an open set inside (X, d_X).

(iii) (Easy) Consider a metric space (X, d). Prove that if $U \subseteq V \subseteq X$, then the interior of U is contained in the interior of V, i.e. $\text{int}(U) \subseteq \text{int}(V)$.

(iv) (Easy) Prove that a metric space (X, d) is connected if and only if there does not exist two sets A and B such that $X = A \cup B$ and their closures are disjoint, i.e. $\text{cl}(A) \cap \text{cl}(B) = \emptyset$.

(v) (Medium) Let $e : [-1, 1] \to \mathbb{R}$ be a continuous function where $e(-1) = -1$ and $e(1) = 1$. Consider the following optimisation problem,

$$\max_{\bar{u} \in \mathbb{R}} \bar{u}$$
$$\text{s.t. } e(\bar{u}) = 0.$$

(This is a simplified version of the Bellman equation connecting the indirect utility function and the expenditure function, which are not examinable.) Prove that there exists a solution, $\bar{u}^*$.

(vi) (Medium) Let U be a connected set inside the metric space (X, d). Prove that the closure of U is connected.

(vii) (Medium) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ and a non-empty non-closed set A such that f is discontinuous at points inside of A and continuous elsewhere. Hint: consider using the indicator function $g(x) = I(x \in \mathbb{Q})$ as a building block.

(viii) (Hard) Prove that there is no continuous injective function $f : [0,1]^2 \to [0,1]$, where both spaces use d_2.

46: Micro 1, May 2022

A natural services company hires workers to do three tasks: field work, desk work and cleaning. Workers are identical. They prefer doing a mix of field work and desk work compared to specialising in one or the other, and they prefer either to cleaning. Workers supply the three types of labour and hire the company to maintain biodiverse habitats around their homes.

(i) Construct a competitive model of the labour and natural services market. *Note: You do not need to model the details of labour preferences specified above, but your model must be general enough to accommodate them.*

(ii) Write down a Bellman equation in which the company chooses output, and its labour choices are buried inside a value function.

(iii) Prove that if the cleaning wage increases, then the firm demands fewer hours of cleaning.

(iv) Suppose the utility functions and production function are strictly increasing and strictly concave. Prove there is at most one equilibrium.

(v) The government worries that cleaning is dangerous, so it proposes banning half the population from cleaning work. (For example, in some parts of India, the Brahmin caste is de facto banned from some types of cleaning.) Assume that in both cases, with and without the ban, there is a unique equilibrium. Prove that

   (a) the prices in the cleaning ban equilibrium are different from the original equilibrium, and

   (b) the people banned from cleaning are made worse off.

47: Skipped.

48: AME, December 2022

Part A

Egypt and Sudan both depend on water from the Nile river. Since the Nile flows through Sudan first, Sudanese households are endowed with all of the water. A firm in each country buys wholesale water and hires local labour, and sells food internationally and retail water locally. Each household supplies labour and wholesale water (Sudan only), and buys food and local retail water. Each firm is owned by the local households.

(i) Formulate a competitive equilibrium model of the international food and wholesale water markets, and the domestic water and labour markets.

(ii) Reformulate the Egyptian firm's profit maximisation problem with a Bellman equation in which all choices except water demand are buried inside a value function.

(iii) Prove that if the wholesale water price goes up, then the Egyption firm uses less water.

Part B

(i) (easy) Consider a metric space (X, d), and two sets U and Y with $U \subseteq Y \subseteq X$. Prove that if U is open in (X, d), then U is open inside (Y, d).

(ii) (easy) Two countries are bargaining over a truce agreement x which can be chosen from a compact metric space (X, d). The countries' utility functions $u_1, u_2 : X \to [0, 1]$ are continuous. Let A be the set of agreements for which country 1 is strictly better off than country 2, i.e. $A = \{x \in X : u_1(x) > u_2(x)\}$. Prove A is open.

(iii) (easy) Find a counter-example to this false conjecture: $\text{int}(\text{cl}(\text{int}(A))) = \text{int}(A)$.

(iv) (easy) Suppose the state space of an infinite horizon dynamic programming problem is $X = \mathbb{R}_{++} \times \{0, 1\}$. Is the metric space of possible value functions, $(B(X), d_\infty)$, a complete metric space?

(v) (medium) Consider any two metric spaces (X, d) and (X, d'). Suppose that for any $x^0 \in X$, the function $f(x) = d'(x, x^0)$ is a continuous function from (X, d) to $(\mathbb{R}_+, d_2)$. Prove that if A is open in (X, d'), then A is open in (X, d).

(vi) (medium) Consider a function $f : X \to Y$ where (X, d_X) and (Y, d_Y) are metric spaces. Consider the metric space $Z = (X \times Y, d_Z)$, where $d_Z(x, y; x', y') = d_X(x, x') + d_Y(y, y')$. Let $A \subseteq Z$ be the set $\{(x, f(x)) : x \in X\}$, which is called the *graph* of f. Prove that if f is continuous then A is closed.

(vii) (medium) Suppose $f : X \to Y$, (X, d_X), (Y, d_Y), (Z, d_Z) and A are defined as in the previous question. Prove that if f is continuous and (X, d_X) is connected, then A is connected. *Hint: Consider the function $g(x) = (x, f(x))$.*

(viii) (hard) Consider the metric spaces (X, d) and (Y, d) where $Y \subseteq X$. Prove that if U is open inside (Y, d), then there exists an open set V inside (X, d) such that $U = V \cap Y$.

49: Micro 1, December 2022

Suppose there are two sources of energy, gas and solar power electricity. For heating homes, the two sources are perfect substitutes. But for manufacturing, they have different uses and are imperfect substitutes. Each household is endowed with gas deposits and solar panels. Households sell solar power directly to each other and to factories, and gas that they sell on the wholesale gas market. Households also sell their labour inelastically. Households buy appliances, electricity and retail gas. Factories use labour, gas and solar power to make appliances. The gas firm uses labour and wholesale gas to make retail gas.

(i) Formulate a competitive model of the wholesale gas, retail gas, solar power, labour and appliance markets.

(ii) Prove that Walras law holds in the context of your model. (Recall: that Walras law says that the market value of the excess demand at market prices is zero, even if those prices do not lead to an equilibrium.) *Hint: substitute the profit functions into the budget constraint.*

(iii) Suppose the factories and the electricity firm merge into a single company. Prove that the merged company's demand for wholesale gas decreases if the price of wholesale gas increases.

(iv) Assume that gas and electricty are normal goods for households, and the utility and production functions are strictly concave. Suppose that there are global warming protests, and that half of the population protest. The protesting households do not sell (or use) any gas. Prove the following:

  (a) There is at most one equilibrium without the protests. (The same logic applies when there are protests.)

  (b) During protests, protestors heat their homes less than non-protestors.

  (c) It is possible to devise a lump-sum tax scheme that makes the protestors heat their homes more, and the non-protestors heat their homes less. (Assume this satisfies the protestors, so they stop protesting.)

50: AME, May 2023

Part A

In the early 19th century, Australia traded mostly with England. Australia exported wool and imported hardware. Households in both countries are endowed with labour. In addition, households in Australia are endowed with wool, and in England are endowed with hardware. Households buy homes and clothes. Homes are made from hardware and labour. Clothes are made from wool and labour.

(i) Formulate a competitive equilibrium model of this economy.

(ii) Suppose all production in Australia is managed by the East India Company's Australian division. Formulate the division's profit maximisation problem in which it chooses its aggregate labour demand first, and the other choices are buried inside a value function.

(iii) Prove that when English wages increase, the demand for English labour decreases.

Part B

(i) (easy) Suppose $f : X \to Y$ and $g : Y \to X$ are Lipshitz continuous of degree $a < 1$. Prove that $h(x) = g(f(x))$ is a contraction of degree a^2.

(ii) (easy) Suppose U and V are open sets inside (X, d_X) and (Y, d_Y) respectively. Prove that $U \times V$ is open inside $(X \times Y, d_Z)$ where $d_Z(x, y; x', y') = d_X(x, x') + d_Y(y, y')$.

(iii) (easy) Let A and B be sets inside (X, d_X) and (Y, d_Y) respectively. Prove that $\mathrm{cl}(A \times B) = \mathrm{cl}(A) \times \mathrm{cl}(B)$ inside $(X \times Y, d_Z)$, where $d_Z(x, y; x', y') = d_X(x, x') + d_Y(y, y')$.

(iv) (medium) Let (X, d_X) and (Y, d_Y) be metric spaces. Suppose $f : X \to X$ is a continuous self-map that has no fixed points. Suppose that there is a bijection g such that $g : X \to Y$ and its inverse $g^{-1} : Y \to X$ are continuous. Prove that (Y, d_Y) has a continuous self-map that has no fixed points.

(v) Consider the following version of Pavoni's (2009) model of unemployment insurance. His notation is as follows (it is unnecessary to answer the questions). U is lifetime utility promised to the unemployed person, U^u is the future promise if the worker remains unemployed tomorrow, U^e is the future promise if the worker finds a job by tomorrow, $W(U^e)$ is the government's value of fulfilling this second promise, π is the probability of finding a job, b is the unemployment payment today, $u(b)$ is the person's utility of receiving a payment of b where $u \in B(\mathbb{R}_+)$, β is the discount rate, and $V(U)$ is the government's value of promising $U \in X \subset \mathbb{R}$ where $V \in B(X)$, which is the solution to the Bellman equation

$$\begin{aligned} V(U) = \sup_{b, U^e, U^u} & -b + \beta[\pi W(U^e) + (1 - \pi)V(U^u)] \\ \text{s.t. } & U = u(b) + \beta[\pi U^e + (1 - \pi)U^u] \\ & \text{and } U^e \geq U^u. \end{aligned}$$

The first constraint says that the government's promise U can be fulfilled by a combination of paying the person today, or making more promises. The second constraint says that the person would prefer to accept all job offers.

(a) (medium) Suppose that the Bellman operator T is a contraction on $(B(X), d_\infty)$. Prove that if u is increasing, then V is (weakly) decreasing.

(b) (medium) Suppose that the Bellman operator T is a contraction on $(CB(X), d_\infty)$ and that the range of u is compact. Prove that there exists an optimal choice of (b, U^e, U^u) for all promises $U \in X$.

(vi) (hard) Let (X, d) be a compact metric space. Suppose $f : X \times [0, 1] \to \mathbb{R}$ is continuous, with distances in the domain measured by

$$d'(x, y; x', y') = d(x, x') + d_2(y, y').$$

Let $g_n : X \to X$ be defined by $g_n(x) = f(x, 1/n)$. Prove that g_n is a convergent sequence inside the metric space $(CB(X), d_\infty)$.

51: Micro 1, May 2023

In the "War of the currents" in the late 1800s, George Westinghouse's and Thomas Edison's companies provided competing electricity distribution systems. Westinghouse's alternating current (AC) can power motors such as refrigerators, whereas Edison's direct current (DC) can power semiconductors such as computers. Rectifiers and inverters can convert between AC and DC. AC won the war, so laptops come with power adapters with rectifiers, but refrigerators do not need inverters.

Suppose Westinghouse owns an AC supplier and Edison owns a DC supplier. (The other firms are held by the rest of the population.) They produce electricity from labour. The first unit needs a lot of labour, so it is inefficient for both firms to operate.

The four eletrical goods – computers, refrigerators, rectifiers and inverters – are produced from labour. Households supply labour inelastically, and buy electrical goods and appropriate electricity to power them. Households derive utility from using computers and refrigerators, which are always turned on.

(i) Formulate a competitive model of the labour, electricity and electrical goods markets (7 in all).

(ii) Prove that either AC or DC wins, i.e. in every equlibrium, either AC or DC are inactive.

(iii) Edison is worried that Westinghouse is going to win, so he proposes a merger of their firms. Write the profit function of the merged firm.

(iv) Edison has a new idea. He could offer a DC package deal that gives the same utility as before. Prove that if the price of inverters increases, then Edison's package includes fewer inverters.

(v) Edison has another idea, lump-sum taxes. Suppose there are both AC and DC equilibria. What lump-sum taxes can the government use to implement the DC equilibrium? Would these improve Edison's DC firm's profits?

52: Micro 1, December 2023

In 1918, the Hotel Saint Gellért in Budapest opened its doors. It was the first hotel to have a swimming pool. The architects (Hegedus, Sebestyen, and Sterk) had to choose the size of the swimming pool, the hotel room size, and the number of hotel rooms. Assume that all hotel rooms are identical. Each household supplies labour to build the hotel, and chooses how long to visit the hotel for. Each household owns an equal share in the hotel.

(i) Formulate a competitive model of the hotel room and labour markets.

Hint 1: there is an infinite number of hotel room markets, one for each type of room (i.e. each combination of room and pool size). But only one is active.

Hint 2: you may assume that households can only demand one type of hotel room (of their choice).

(ii) Is it possible to normalise prices by dividing by the price of a type of hotel room that is not traded?

(iii) The Société de l'Art Nouvea decided that the workers aren't working hard enough. Is it possible to design lump sum taxes to increase the aggregate hours worked?

(iv) Prove that if the hotel room price increases (i.e. the price of the type of hotel room that the hotel supplies in equilibrium), then the hotel builds extra rooms.

(v) Suppose that at market prices (possibly non-equilibrium prices), all hotel room markets clear. Prove that the labour market clears.

Note: you will get more points if you show the details about how to adapt the logic from lectures.

53: AME, December 2023

Part A

Suppose an engineering firm designs and builds apartment buildings. It hires both full-time and part-time workers. Assume that full-time workers are more productive, because they complete urgent tasks more quickly, and are easier to reach to resolve problems. Assume that all households have two workers, and some households have children. Households with children have a stronger preference for part-time work. Households own the engineering firm, supply labour, and buy homes.

(i) Formulate a competitive model of the three markets (the market for apartments, and full-time and part-time labour markets).

(ii) Prove that if the wages of part-time workers increases, then firm demands fewer part-time workers.

(iii) Reformulate the firm's problem with a Bellman equation in which the only choice is the amount of apartment construction.

Part B

(i) (easy) Provide a counter-example to the following false claim: If (X, d) is a metric space, and the interior of $A \subset X$ is connected, then A is connected.

(ii) (easy) Consider the metric spaces (X, d_X), (Y, d_Y) and (Z, d_Z) where $Z = X \times Y$ and $d_Z(x, y; x', y') = d_X(x, x') + d_Y(y, y')$. Prove that if (Z, d_Z) is connected, then (X, d_X) is connected.

(iii) (easy) Pick any set A inside a metric space (X, d). Pick any radius $r > 0$ and let $U = \{x : (x, a) \in X \times A, d(x, a) < r\}$ be the set of all points in X that have a distance of less than r to some point inside A. Prove that U is an open set.

(iv) (easy) Suppose there are two bidders in an auction for the remnants of a bankrupt car factory. The first bidder values the factory at £20m. The first bidder spied on the second bidder, and knows he will bid £10m. Thus, his (expected) profit when bidding b million is

$$\pi(b) = \begin{cases} 0 & \text{if } b < 10, \\ 5 & \text{if } b = 10, \\ 20 - b & \text{if } b > 10. \end{cases}$$

Calculate the range $\pi(\mathbb{R})$, the maximum $\max \pi(\mathbb{R})$ and the supremum $\sup \pi(\mathbb{R})$, or prove that they do not exist.

(v) (medium) Suppose that (X, d) is unbounded. Prove that there is a continuous function $f : X \to \mathbb{R}$ that does not have a maximum.

(vi) (medium) Suppose (X, d) is a disconnected metric space. Prove that there is a continuous function $f : X \to X$ that does not have any fixed point, i.e. there is no $x^* \in X$ with $f(x^*) = x^*$.

(vii) (medium) Consider a contraction $f : X \to X$ of degree k on the metric space (X, d). Let $A_0 = B_{r_0}(x_0)$ be an open ball, and let $A_{n+1} = f(A_n)$. Prove that A_n is contained in a ball of radius $r_n = r_0 k^n$.

(viii) (hard) Suppose that $A \subseteq U$ and $B \subseteq V$ are non-empty sets, and U and V are disjoint open sets, and all four sets lie inside the metric space (X, d). Prove that $A \cup B$ is disconnected.